Sharp criteria of Liouville type for some nonlinear systems

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Abstract

In this paper, we establish the sharp criteria for the nonexistence of positive solutions to the Hardy-Littlewood-Sobolev (HLS) type system of nonlinear equations and the corresponding nonlinear differential systems of Lane-Emden type equations. These nonexistence results, known as Liouville type theorems, are fundamental in PDE theory and applications. A special iteration scheme, a new shooting method and some Pohozaev type identities in integral form as well as in differential form are created. Combining these new techniques with some observations and some critical asymptotic analysis, we establish the sharp criteria of Liouville type for our systems of nonlinear equations. Similar results are also derived for the system of Wolff type integral equations and the system of γ-Laplace equations. A dichotomy description in terms of existence and nonexistence for solutions with finite energy is also obtained.

Keywords: critical exponents, Liouville type theorems, HLS type integral equations, Wolff type integral equations, semilinear Lane-Emden equations, γ-Laplace equations, necessary and sufficient conditions of existence/nonexistence.
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1 Introduction

In this paper, we establish sharp criteria for existence and nonexistence of positive solutions to the Hardy-Littlewood-Sobolev (HLS) system of nonlinear equations

$$\left\{\begin{array}{l} u(x)=\displaystyle\int_{R^n}\frac{v^q(y)dy}{|x-y|^{n-\alpha}} \\ v(x)=\displaystyle\int_{R^n}\frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \end{array}\right. \tag{1.1}$$

and the corresponding nonlinear differential systems of Lane-Emden type equations

$$\left\{\begin{array}{l} (-\Delta)^k u=v^q,\ \ u,v>0, \\ (-\Delta)^k v=u^p,\ \ p,q>0. \end{array}\right. \tag{1.2}$$

These systems are the ‘blow up’ equations for a large class of systems of nonlinear equations arising from geometric analysis, fluid dynamics, and other physical

sciences. The nonexistence of positive solutions for systems of 'blow up' type like (1.1) and (1.2), known as Liouville type theorem, is useful in deriving existence, a priori estimate, regularity and asymptotic analysis of solutions. Another important topic is the study of the Wolff type system of nonlinear equations:

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x) \\ v(x) = W_{\beta,\gamma}(u^p)(x), \end{cases} \tag{1.3}$$

and the corresponding system of γ-Laplace equations

$$\begin{cases} -\Delta_\gamma u(x) = v^q(x), \quad x \in R^n, \\ -\Delta_\gamma v(x) = u^p(x), \quad x \in R^n. \end{cases} \tag{1.4}$$

Recall a Liouville-type theorem for the Lane-Emden equation

$$-\Delta u = u^p, \quad \textit{in } R^n \ (n \geq 3) \tag{1.5}$$

obtained by Caffarelli, Gidas and Spruck [1]: if $p \in (0, \frac{n+2}{n-2})$, then (1.5) has no positive classical solution. When $p \geq \frac{n+2}{n-2}$, (1.5) has positive classical solution. Namely, the right end point $\frac{n+2}{n-2}$ is a sharp criterion distinguishing the existence and the nonexistence. Numbers like this, separating the existence and the nonexistence, are called the critical exponents.

In the following theorem, we obtain the sharp criteria on the existence and the nonexistence of solutions to (1.2):

Theorem 1.1. *Assume $k \in [1, \frac{n}{2})$ is an integer.*

(1) The $2k$-order equation

$$(-\Delta)^k u(x) = u^p(x), \quad x \in R^n \tag{1.6}$$

has positive solutions if and only if $p \geq \frac{n+2k}{n-2k}$.

(2) The $2k$-order system (1.2) has a pair of positive solutions (u, v) if $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n}$.

Remark 1.1.

1. When $k = 1$, the first result is coincident with the result in [1].
2. Part 2 of our Theorem 1.1 together with the nonexistence result of Souplet [48] imply that: for $n \leq 4$, (1.2) has a pair of solutions *if and only if* $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n}$.
3. For $k = 1$, the nonexistence of solutions to (1.2), known as the Lane-Emden conjecture, is still open for $n \geq 5$ (cf. [48]). In the non-subcritical case, i.e. $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}$, (1.2) with $k = 1$ has the positive solutions (cf. [46]).

Next, we consider the HLS type system of nonlinear equations (1.1) and its scalar case $v \equiv u$, $q = p$:

$$u(x) = \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}. \tag{1.7}$$

Such equations are related to the study of the best constant of Hardy–Littlewood-Soblev (HLS) inequality. Lieb [35] classified all the extremal solutions of (1.7), and thus obtained the best constant in the HLS inequalities. He posed the classification of all the solutions of (1.7) as an open problem.

The corresponding PDE is the semilinear equation involving a fractional order differential operator

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}, \quad u > 0, \quad \text{in } R^n. \tag{1.8}$$

The classification of the solutions of (1.8) with $\alpha = 2$ has provided an important ingredient in the study of the prescribing scalar curvature problem. It is also essential in deriving priori estimates in many related nonlinear elliptic equations. It was well studied by Gidas, Ni, and Nirenberg [14]. They proved that all the positive solutions with reasonable behavior at infinity, namely

$$u(x) = O(\frac{1}{|x|^{n-2}}) \tag{1.9}$$

are radially symmetric about some point. Caffarelli, Gidas, and Spruck removed the decay condition (1.9) and obtained the same result (cf. [1]). Then Chen and Li [5], and Li [28] simplified their proofs. Later, Chang, Yang and Lin also considered some higher order equations (cf. [3], [36]). Wei and Xu [50] generalized this result to the solutions of more general equation (1.8) with α being any even numbers between 0 and n. Chen, Li, and Ou solved the open problem as stated for the integral equation (1.7) or the corresponding PDE (1.8) in [11]. The unique class of solutions can assume the form

$$u(x) = c(\frac{t}{t^2 + |x|^2})^{\frac{n-\alpha}{2}}. \tag{1.10}$$

Other related work can be seen in [4], [31] and [32].

Chen, Li and Ou [10] introduced the method of moving planes in integral forms to study the symmetry of the solutions for the HLS type system (1.1). Jin-Li and Hang thoroughly discussed the regularity of the solutions of (1.1) (cf. [18] and [20]). They found the optimal integrability intervals and established the smoothness for the integrable solutions. Based on the results, [27] gave the asymptotic behavior of the integrable solutions when $|x| \to 0$ and $|x| \to \infty$. Some Liouville type results can be seen in [2] and [6].

Another significance of the work [10] is the equivalence of the integral equations and the PDEs involving the fractional order differential operator. Recently, the fractional Laplacians were applied extensively to describe various physical and finance phenomena, such as anomalous diffusion, turbulence and

water waves, molecular dynamics, relativistic quantum mechanics, and stable Levy process. The equivalence provides a technique in studying the PDEs: one can use the corresponding integral equations to investigate the global properties for those phenomena.

A positive solution u of (1.7) is called a *finite energy solution*, if $u \in L^{p+1}(R^n)$. Similarly, positive solutions u, v are called finite energy solutions of (1.1), if $u \in L^{p+1}(R^n)$, $v \in L^{q+1}(R^n)$. Now, we point out the relation between the critical conditions and the existence of finite energy solutions of (1.7) and (1.1).

Theorem 1.2. *(1) The HLS type integral equation (1.7) has positive solutions in $L^{p+1}(R^n)$ if and only if*

$$p = \frac{n+\alpha}{n-\alpha}. \tag{1.11}$$

(2) The system (1.1) has a pair of positive solutions (u, v) in $L^{p+1}(R^n) \times L^{q+1}(R^n)$ if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}. \tag{1.12}$$

Corollary 1.3. *Let $k \in [1, n/2)$ be an integer.*

(1) Assume $p > 1$. The $2k$-order PDE (1.6) has positive solutions in $L^{p+1}(R^n)$ if and only if

$$p = \frac{n+2k}{n-2k}. \tag{1.13}$$

(2) Assume $pq > 1$. System (1.2) has a pair of positive solutions (u, v) in $L^{p+1}(R^n) \times L^{q+1}(R^n)$ if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2k}{n}. \tag{1.14}$$

Remark 1.2.

1. In the subcritical case $p < \frac{n+\alpha}{n-\alpha}$, Theorem 3 in [9] shows that (1.7) has no locally finite energy solution by using the method of moving planes and the Kelvin transformation. For system (1.1), the proof of nonexistence in the subcritical case $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\alpha}{n}$ is rather difficult. It is usually called the HLS conjecture (cf. [2] and [6]). Partial results are known.

   (i) If $p \leq \frac{\alpha}{n-\alpha}$ or $q \leq \frac{\alpha}{n-\alpha}$, (1.1) has no any positive solution. In addition, if $p = 1$ or $q = 1$, then (1.1) has no any positive solution. If the subcritical condition $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$ holds, then (1.2) with $k = 1$ has no locally bounded positive solution (u, v) in $L^{p_1}(R^n) \times L^{q_1}(R^n)$, where $p_1 = \frac{n(pq-1)}{2(q+1)}$, $q_1 = \frac{n(pq-1)}{2(p+1)}$ (cf. [6]).

   (ii) If $\alpha \in [2, n)$, (1.1) has no any radial positive solution (cf. [2]). If $pq > 1$, (1.2) has no any radial positive solution (cf. [37]).

2. In the supercritical case $p > \frac{n+\alpha}{n-\alpha}$ with $\alpha = 2$, Li, Ni and Serrin proved the semilinear Lane-Emden equation (1.5) has the decay solution (cf. [32], [40] and [42]). According to Corollary 1.3, the energy of those solutions u are infinite. Namely, $\|u\|_{p+1} = \infty$. Similarly, the positive solutions u, v obtained in [46] when $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}$ are not the finite energy solutions (i.e. $\|u\|_{p+1} = \|v\|_{q+1} = \infty$).

3. Theorem 1.2 shows that the critical conditions are the sufficient and necessary conditions of existences of the finite energy solutions for the HLS type integral equation and the system. On the other hand, if the critical conditions hold, we want to know whether all the positive classical solutions u, v are finite energy solutions. Namely, $\|u\|_{p+1}, \|v\|_{q+1} < \infty$. For the scalar equation (1.7) with the critical case $p = \frac{n+\alpha}{n-\alpha}$, (1.10) is the unique class of finite energy solutions of (1.7) (cf. [11]). For system (1.1), it is still open.

The following γ-Laplace equation is also concerned in this paper

$$-\Delta_\gamma u(x) := -div(|\nabla u|^{\gamma-2}\nabla u) = u^p(x), \quad x \in R^n. \tag{1.15}$$

Theorem 1.4. *The γ-Laplace equation (1.15) has positive classical solutions with $\int_{R^n} |\nabla u|^\gamma dx < \infty$ if and only if $p = \gamma^* - 1$, where $\gamma^* = \frac{n\gamma}{n-\gamma}$.*

Remark 1.3. Serrin and Zou [47] proved (1.15) has the classical solution u if and only if $p \geq \gamma^* - 1$. Furthermore, Theorem 1.4 shows that u is also a finite energy solution (i.e. $\nabla u \in L^\gamma(R^n)$, see Theorem 4.7) if $p = \gamma^* - 1$, and u is a infinite energy solution if $p > \gamma^* - 1$.

To study the γ-Laplace equations, we introduce the Wolff potential of a positive locally integrable function f

$$W_{\beta,\gamma}(f)(x) := \int_0^\infty [\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

The integral equation involving the Wolff potential

$$u(x) = c(x) W_{\beta,\gamma}(u^p)(x) \tag{1.16}$$

is related with the study of many nonlinear problems. The Wolff potentials are helpful to understand the nonlinear PDEs such as the γ-Laplace equation and the k-Hessian equation (cf. [23], [25], [26] and [44]). According to [21], if $\inf_{R^n} u = 0$, there exists $C > 0$ such that the positive solution u of (1.15) satisfies

$$\frac{1}{C} W_{1,\gamma}(u^p)(x) \leq u(x) \leq C W_{1,\gamma}(u^p)(x), \quad x \in R^n.$$

Thus, u solves (1.16) for some double bounded $c(x)$. Here, a function $c(x)$ is called *double bounded*, if there exists a positive constant $C > 0$ such that

$$\frac{1}{C} \leq c(x) \leq C, \quad \forall\ x \in R^n.$$

For the coupling system

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x) \\ v(x) = W_{\beta,\gamma}(u^p)(x), \end{cases}$$

Chen and Li [7] proved the radial symmetry for the integrable solutions. Afterward, Ma, Chen and Li [38] used the regularity lifting lemmas to obtain the optimal integrability and the Lipschitz continuity. Based on these results, [24] obtained the decay rates of the integrable solutions when $|x| \to \infty$.

The critical exponents and the critical conditions play a key role under the scaling transform. In the following, some interesting observations are listed.

Under the scaling transform $u_\mu(x) = \mu^\sigma u(\mu x)$,

(1) the HLS equation (1.7) and the energy $\|u\|_{p+1}$ *are invariant if and only if* $p = \frac{n+\alpha}{n-\alpha}$;

(2) the Wolff equation

$$u(x) = W_{\beta,\gamma}(u^p)(x) \tag{1.17}$$

and the energy $\|u\|_{p+\gamma-1}$ *are invariant if and only if* $p = \frac{n+\beta\gamma}{n-\beta\gamma}(\gamma-1)$; *(1.17) and the energy* $\|u\|_{p+1}$ *are invariant if and only if* $p = \gamma^* - 1$ *with* $\gamma^* = \frac{n\gamma}{n-\beta\gamma}$;

(3) the γ*-Laplace equation (1.15) and the energy* $\|u\|_{p+\gamma-1}$ *are invariant if and only if* $p = \frac{n+\gamma}{n-\gamma}(\gamma-1)$; *(1.15) and the energy* $\|u\|_{p+1}$ *are invariant if and only if* $p = \gamma^* - 1$ *with* $\gamma^* = \frac{n\gamma}{n-\gamma}$;

(4) the HLS system (1.1) and $\|u\|_{p+1}$, $\|v\|_{q+1}$ *are invariant if and only if* $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}$;

(5) the Wolff system (1.3) and $\|u\|_{p+\gamma-1}$, $\|v\|_{q+\gamma-1}$ *are invariant if and only if* $\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\beta\gamma}{n(\gamma-1)}$; *(1.3) and* $\|u\|_{p+1}$, $\|v\|_{q+1}$ *are invariant if and only if* $p = q$ *or* $\gamma = 2$;

(6) the γ*-Laplace system (1.4) and* $\|u\|_{p+\gamma-1}$, $\|v\|_{q+\gamma-1}$ *are invariant if and only if* $\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\gamma}{n(\gamma-1)}$; *(1.4) and* $\|u\|_{p+1}$, $\|v\|_{q+1}$ *are invariant if and only if* $p = q$ *or* $\gamma = 2$.

Remark 1.4. Here an interesting observation is, the critical exponent $\frac{n+\gamma}{n-\gamma}(\gamma-1)$ is different from the divided exponent γ^*-1 in Theorem 1.4 except $\gamma = 2$. The reason is that those critical numbers are in the different finite energy functions classes $L^{p+\gamma-1}(R^n)$ and $L^{p+1}(R^n)$, respectively.

Next, we are concerned with the sufficient and necessary conditions for the existence of the positive solutions of equations and systems with some double bounded coefficients. Here, new divided numbers and conditions appear.

Theorem 1.5. *(1) The equation*

$$u(x) = c(x) \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \tag{1.18}$$

has positive solutions for some double bounded $c(x)$ *if and only if* $p > \frac{n}{n-\alpha}$.

(2) The HLS system

$$\begin{cases} u(x) = c_1(x) \displaystyle\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} \\ v(x) = c_2(x) \displaystyle\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}. \end{cases} \tag{1.19}$$

has positive solutions u, v for some double bounded $c_1(x)$ and $c_2(x)$, if and only if $pq > 1$ and $\max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\} < n - \alpha$.

Corollary 1.6. *Let $k \in [1, n/2)$ be an integer.*

(1) Assume $p > 1$. The $2k$-order PDE

$$(-\Delta)^k u(x) = c(x)u^p(x), \quad x \in R^n, \tag{1.20}$$

has positive solutions for some double bounded $c(x)$ if and only if $p > \frac{n}{n-2k}$.

(2) Assume $pq > 1$. The system

$$\begin{cases} (-\Delta)^k u(x) = c_1(x)v^q(x) \\ (-\Delta)^k v(x) = c_2(x)u^p(x). \end{cases} \tag{1.21}$$

has positive solutions u, v for some double bounded $c_1(x)$ and $c_2(x)$, if and only if $\max\{\frac{2k(p+1)}{pq-1}, \frac{2k(q+1)}{pq-1}\} < n - 2k$.

Theorem 1.7. *(1) The equation (1.16)*

$$u(x) = c(x)W_{\beta,\gamma}(u^p)(x)$$

has positive solutions for some double bounded $c(x)$, if and only if

$$p > \frac{n(\gamma-1)}{n-\beta\gamma}.$$

(2) The system

$$\begin{cases} u(x) = c_1(x)W_{\beta,\gamma}(v^q)(x) \\ v(x) = c_2(x)W_{\beta,\gamma}(u^p)(x). \end{cases} \tag{1.22}$$

has positive solutions u, v for some double bounded $c_1(x)$ and $c_2(x)$, if and only if $pq > (\gamma-1)^2$ and

$$\max\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\} < \frac{n-\beta\gamma}{\gamma-1}.$$

Corollary 1.8. *(1) If $p > \frac{n(\gamma-1)}{n-\gamma}$, then*

$$-\Delta_\gamma u(x) = c(x)u^p(x), \quad x \in R^n \tag{1.23}$$

has positive solutions for some double bounded $c(x)$. If $0 < p \le \frac{n(\gamma-1)}{n-\gamma}$, then for any double bounded $c(x)$, (1.23) has no positive solution satisfying $\inf_{R^n} u = 0$.

(2) If $pq > (\gamma-1)^2$ and $\max\{\frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}\} < \frac{n-\gamma}{\gamma-1}$, then there exist positive solutions u, v of the γ-Laplace system

$$\begin{cases} -\Delta_\gamma u(x) = c_1(x) v^q(x), & x \in R^n, \\ -\Delta_\gamma v(x) = c_1(x) u^p(x), & x \in R^n \end{cases} \tag{1.24}$$

for some double bounded $c_1(x)$ and $c_2(x)$. On the contrary, for any double bounded functions $c_1(x)$ and $c_2(x)$, if one of the following conditions holds

(i) $0 < pq \leq (\gamma-1)^2$;

(ii) $pq > (\gamma-1)^2$ and

$$\max\{\frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}\} \geq \frac{n-\gamma}{\gamma-1}.$$

Then (1.24) has no positive solutions u, v satisfying $\inf_{R^n} u = \inf_{R^n} v = 0$.

Remark 1.5. Comparing with Theorem 1.2-Theorem 1.4, we obtain, from Theorem 1.5-Corollary 1.8, other divided conditions on the existence of the positive solutions of the equations and systems with ratio coefficients $c(x)$, $c_1(x)$ and $c_2(x)$. These divided conditions are called *the secondary critical conditions*. The secondary critical conditions are more relaxed than those in Theorem 1.2 -Theorem 1.4 because the solutions classes of the equations and systems with ratio coefficients are larger than that in the case of $c(x) \equiv Constant$.

In the proofs of Theorem 1.5-Corollary 1.8, we apply a special iteration scheme and some critical asymptotic analysis to establish the existence and the nonexistence, and hence obtain the sharp criteria.

The contents of this paper are as follows. In Section 2, we prove Theorem 1.5 (1), Corollary 1.6 (1), Theorem 1.7 (1) and Corollary 1.8 (1). Theorem 1.5 (2), Corollary 1.6 (2), Theorem 1.7 (2) and Corollary 1.8 (2) are proved in Section 3. In Section 4.2, we prove (1) of Theorem 1.2, which covers (1) of Corollary 1.3. The proof of of Theorem 1.4 is given in Section 4.3. In Section 5, we give the proofs of (2) of Theorem 1.2 and (2) of Corollary 1.3. The argument on Theorem 1.1 is given in Sections 6 (see Remark 6.1).

2 Equations with variable coefficients

The following proposition is often used in this paper.

Proposition 2.1. *If $w \in L^1(R^n)$, then we can find $R_j \to \infty$ such that*

$$R_j \int_{\partial B_{R_j}(0)} |w| ds \to 0 \quad \text{and} \quad R_j^n \int_{S^{n-1}} |w| ds \to 0.$$

Here $S^{n-1} = \partial B_1(0)$.

Proof. In view of $\|w\|_{L^1(R^n)} < \infty$, it follows from the definition of the improper integral that

$$\lim_{R\to\infty} \int_{B_{2R}(0)\setminus B_R(0)} |w|dx = 0.$$

Hence, as $R \to \infty$,

$$\inf_{[R,2R]} (r \int_{\partial B_r(0)} |w|ds) \to 0 \quad and \quad \inf_{[R,2R]} (r^n \int_{S^{n-1}} |w|ds) \to 0.$$

There exist $R_j \in [R, 2R]$, such that as $R_j \to \infty$,

$$R_j \int_{\partial B_{R_j}(0)} |w|ds \to 0 \quad and \quad R_j^n \int_{S^{n-1}} |w|ds \to 0.$$

$\square$

2.1 HLS type integral equation

In this subsection, we give a relation between the exponents and the existence of positive solutions for integral equations involving the Riesz potentials. First we consider the semilinear Lane-Emden type equations

$$-\Delta u(x) = c(x)u^p(x), \quad x \in R^n. \tag{2.1}$$

Theorem 2.2. *Let $p \geq 1$. Then (2.1) has a positive solution for some double bounded $c(x)$, if and only if $p > \frac{n}{n-2}$.*

Proof. Step 1. If $p > \frac{n}{n-2}$, we claim that (2.1) has the special solution as follows

$$u(x) = \frac{1}{(1+|x|^2)^\theta}, \tag{2.2}$$

where $\theta > 0$ will be determined later.

Denote $|x|$ by r, and set $U(r) = U(|x|) = u(x)$. By a simply calculation, we obtain

$$-\Delta u = -U_{rr} - \frac{n-1}{r}U_r = \frac{2\theta}{(1+r^2)^{\theta+1}} \frac{(n-2-2\theta)r^2 + n}{1+r^2}. \tag{2.3}$$

Take $\theta = \frac{1}{p-1}$. Then $n - 2 - 2\theta > 0$ and

$$-\Delta u = \frac{c(r)}{(1+r^2)^{\theta+1}} = c(r)u^{1+1/\theta} = c(r)u^p.$$

Namely, (2.2) with the slow rate $2\theta = \frac{2}{p-1}$ is a solution for some double bounded $c(r)$.

Moreover, if $p = \frac{n+2}{n-2}$, there also exists a fast decaying solution with rate $2\theta = n-2$. Now,

$$-\Delta u = \frac{c(r)}{(1+r^2)^{\theta+2}} = c(r)u^{1+2/\theta} = c(r)u^p.$$

Namely, (2.2) with the fast rate $2\theta = n-2$ is a solution for some double bounded $c(r)$.

Step 2. We prove (2.1) has no positive solution when $1 \leq p \leq \frac{n}{n-2}$.

Otherwise, let u be a positive solution. Take $x_0 \in R^n$ and denote $B_R(x_0)$ by B. Let

$$\phi(x) = \phi_R(x) = c_R\left(\frac{1}{|x-x_0|^{n-2}} - \frac{1}{R^{n-2}}\right),$$

where $0 < c_R \to c_* \in (0,\infty)$ as $R \to \infty$. Then, ϕ solves

$$\begin{cases} -\Delta\phi(x) = \delta(x), & x \in B, \\ \phi = 0, & on\ \partial B. \end{cases}$$

Here δ is a Dirac function at x_0. Then,

$$\begin{aligned} &\int_B c(x)u^p(x)\phi(x)dx = -\int_B \phi\Delta u dx = \int_B \nabla u\nabla\phi dx \\ &= \int_{\partial B} u\partial_\nu\phi ds - \int_B u\Delta\phi dx = \int_{\partial B} u\partial_\nu\phi ds + u(x_0). \end{aligned} \tag{2.4}$$

Here ν is the unit outward normal vector on ∂B. Noting $\partial_\nu\phi < 0$, we have

$$\int_B u^p\phi dx \leq cu(x_0) < \infty.$$

Let $R \to \infty$, there holds

$$\int_{R^n} \frac{u^p(y)dy}{|x_0-y|^{n-2}} < \infty.$$

According to Proposition 2.1, there exists $R_j \to \infty$ (we still denote it by R) such that

$$R\int_{\partial B} \frac{u^p(y)ds}{R^{n-2}} \to 0.$$

Thus, noting $p \geq 1$, we can use the Hölder inequality to deduce that

$$\left|\int_{\partial B} u\partial_\nu\phi ds\right| \leq \frac{c}{R^{n-1}}\int_{\partial B} uds \leq \frac{c}{R^{2/p}}\left(R\int_{\partial B} \frac{u^p(y)ds}{R^{n-2}}\right)^{1/p} \to 0,$$

when $R \to \infty$. Let $R \to \infty$ in (2.4), then

$$u(x_0) = c\int_{R^n} \frac{c(y)u^p(y)dy}{|x_0-y|^{n-2}}.$$

Write $w(x) = c^{1/p}(x)u(x)$, then w solves the integral equation involving the Newton potential

$$w(x) = c^{1/p}(x) \int_{R^n} \frac{w^p(y)dy}{|x-y|^{n-2}}.$$

However, this integral equation has no positive solution for any double bounded c when $0 < p \le \frac{n}{n-2}$. The proof is a special case of the corresponding proof of Theorem 2.3, which handles a more general integral equation involving the Riesz potential. □

Theorem 2.3. *The HLS type integral equation*

$$u(x) = c(x) \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \tag{2.5}$$

has a positive solution for some double bounded $c(x)$, if and only if

$$p > \frac{n}{n-\alpha}. \tag{2.6}$$

Proof. When $|x| \le 2R$ for some $R > 0$, $u(x)$ is proportional to $\int_{R^n} |x - y|^{\alpha-n} u^p(y)dy$. Thus, we only consider the case of $|x| > 2R$.

Step 1. Inserting (2.2) into the right hand side of (2.5), we can find some double bounded function $c(x)$ such that as $|x| > 2R$ for some $R > 0$,

$$\begin{aligned}\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} &= \frac{c(x)}{(1+|x|^2)^{(n-\alpha)/2}} \int_{B_R(0)} \frac{dy}{(1+|y|^2)^{p\theta}} \\ &+ \frac{c(x)}{(1+|x|^2)^{p\theta}} \int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}} \\ &+ c(x) \int_{B_R^c(0)\setminus B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}|y|^{2p\theta}}.\end{aligned}$$

If $p > \frac{n}{n-\alpha}$, we take $2\theta = \frac{\alpha}{p-1}$ and hence $\alpha < 2p\theta < n$. Then,

$$\begin{aligned}&\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \\ &= \frac{c(x)}{(1+|x|^2)^{(n-\alpha)/2}} + \frac{c(x)}{(1+|x|^2)^{p\theta-\alpha/2}} + c(x) \int_{|x|/2}^{\infty} r^{n-(n-\alpha+2p\theta)} \frac{dr}{r} \\ &= \frac{c(x)}{(1+|x|^2)^{p\theta-\alpha/2}} = c(x)u(x)\end{aligned}$$

for some double bounded function $c(x)$. This result shows that (2.5) has the slowly decaying radial solution as (2.2).

Moreover, we can also find a fast decaying solution. Now, take $2\theta = n - \alpha$,

then $2p\theta > n$ as long as $p > \frac{n}{n-\alpha}$. Thus,

$$\begin{aligned}
&\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \\
&= \frac{c(x)}{(1+|x|^2)^{(n-\alpha)/2}} + \frac{c(x)}{(1+|x|^2)^{p\theta-\alpha/2}} + c(x)\int_{|x|/2}^{\infty} r^{n-(n-\alpha+2p\theta)}\frac{dr}{r} \\
&= \frac{c(x)}{(1+|x|^2)^{(n-\alpha)/2}} = c(x)u(x)
\end{aligned}$$

for some double bounded function $c(x)$.

Step 2. We prove (2.5) has no positive solution when $0 < p < \frac{n}{n-\alpha}$.

Suppose u is a positive solution, then it follows a contradiction. In fact, when $|x| > R$ with $R > 0$, $|x-y| \le 2|x|$ for $y \in B_R(0)$. In addition, $\int_{B_R(0)} u^p(y)dy \ge c$. Hence,

$$u(x) \ge c|x|^{\alpha-n}\int_{B_R(0)} u^p(y)dy \ge \frac{c}{|x|^{a_0}}, \quad for\ |x| > R.$$

Here $a_0 = n-\alpha$. Using this estimate, for $|x| > R$ we also get

$$u(x) \ge c\int_{B_{|x|/2}(x)} \frac{|y|^{-pa_0}dy}{|x-y|^{n-\alpha}} \ge \frac{c}{|x|^{pa_0-\alpha}} := \frac{c}{|x|^{a_1}}.$$

By induction, we can obtain

$$u(x) \ge \frac{c}{|x|^{a_j}}, \quad for\ |x| > R,$$

where $j = 0, 1, \cdots$, and

$$a_j = pa_{j-1} - \alpha.$$

In view of $0 < p < \frac{n}{n-\alpha}$, we claim that $\{a_j\}$ is decreasing. In fact, $a_1 - a_0 = (p-1)a_0 - \alpha = (p-1)(n-\alpha) - \alpha < 0$. Suppose $a_k < a_{k-1}$ for $k = 1, 2, \cdots, j$, then

$$a_{j+1} - a_j = (p-1)a_j - \alpha < (p-1)a_0 - \alpha = (p-1)(n-\alpha) - \alpha < 0.$$

This induction shows our claim.

Next we claim that there exists j_0 such that

$$a_{j_0} < 0. \tag{2.7}$$

Once it is verified, then

$$u(x) \ge c\int_{R^n \setminus B_R(0)} \frac{|y|^{-pa_{j_0}}dy}{|x-y|^{n-\alpha}} = \infty.$$

It is impossible.

Proof of (2.7). In fact,

$$\begin{aligned}
a_j &= pa_{j-1} - \alpha = p(pa_{j-2} - \alpha) - \alpha = \cdots \\
&= p^j a_0 - \alpha(p^{j-1} + p^{j-2} + \cdots + p + 1).
\end{aligned}$$

When $p \in (1, \frac{n}{n-\alpha})$,

$$a_j = p^j(n-\alpha) - \alpha\frac{p^j - 1}{p-1} = (n - \alpha - \frac{\alpha}{p-1})p^j + \frac{\alpha}{p-1}.$$

By virtue of $p < \frac{n}{n-\alpha}$, $n - \alpha - \frac{\alpha}{p-1} < 0$, we can find a suitably large j_0 such that $a_{j_0} < 0$.

When $p = 1$, $a_j = a_0 - \alpha j$. Thus, $a_{j_0} < 0$ for some suitably large j_0.

When $p \in (0, 1)$, let $j \to \infty$. Then

$$a_j = p^j a_0 - \alpha\frac{1 - p^j}{1-p} \to -\frac{\alpha}{1-p} < 0.$$

This implies $a_{j_0} < 0$ for some j_0.

Thus, (2.7) is verified.

Step 3. We prove (2.5) has no positive solution when $p = \frac{n}{n-\alpha}$.

Otherwise, u is a positive solution. For $R > 0$, denote $B_R(0)$ by B. From (2.5) it follows that

$$u(x) \geq \frac{1}{(R + |x|)^{n-\alpha}} \int_B u^p(y)dy. \tag{2.8}$$

Thus, taking p powers of (2.8) and integrating on B, we have

$$\begin{aligned} \int_B u^p(x)dx \quad &\geq \int_B \frac{dx}{(R+|x|)^n}(\int_B u^p(y)dy)^p \\ &\geq c(\int_B u^p(y)dy)^p. \end{aligned} \tag{2.9}$$

Here c is independent of R. Letting $R \to \infty$, we see $u \in L^p(R^n)$.

Taking p powers of (2.8) and integrating on $A_R := B_{2R}(0) \setminus B_R(0)$, we get

$$\int_{A_R} u^p(x)dx \geq \int_{A_R} \frac{dx}{(R+|x|)^n}(\int_B u^p(y)dy)^p \geq c(\int_B u^p(y)dy)^p.$$

Letting $R \to \infty$, and noting $u \in L^p(R^n)$, we obtain

$$\int_{R^n} u^p(y)dy = 0,$$

which contradicts with $u > 0$. □

Corollary 2.4. *Assume $k \in [1, n/2)$ and $p > 1$. The higher order semilinear PDE*

$$(-\Delta)^k u(x) = c(x)u^p(x), \quad x \in R^n, \tag{2.10}$$

has a positive solution for some double bounded $c(x)$, if and only if $p > \frac{n}{n-2k}$.

Proof. If $u > 0$ solves the integral equation (2.5) with $\alpha = 2k$, it is easy to see that u also solves the higher order semilinear PDE (2.10). On the contrary, if $p > 1$ and u solves (2.10), [37] proved $(-\Delta)^i u > 0$ for $i = 1, 2, \cdots, k-1$. Similar to the argument in [11], (2.10) is equivalent to (2.5) with $\alpha = 2k$. Therefore, if $p > 1$, Theorem 2.3 shows that (2.10) has positive solutions for some double bounded function $c(x)$, if and only if $p > \frac{n}{n-2k}$. □

2.2 Integral equation involving the Wolff potential

Theorem 2.5. *The Wolff type integral equation*

$$u(x) = c(x)W_{\beta,\gamma}(u^p)(x) \tag{2.11}$$

has a positive solution for some double bounded $c(x)$*, if and only if*

$$p > \frac{n(\gamma-1)}{n-\beta\gamma}.$$

Proof. Step 1. Existence.

Inserting (2.2) into $W_{\beta,\gamma}(u^p)(x)$, we obtain

$$W_{\beta,\gamma}(u^p)(x) = (\int_0^{|x|/2} + \int_{|x|/2}^{\infty})[\int_{B_t(x)} \frac{dy}{(1+|y|^2)^{p\theta}} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} := I_1 + I_2.$$

When $|x| \leq R$ for some $R > 0$, then u is proportional to $W_{\beta,\gamma}(u^p)$. So we also only consider suitably large $|x|$.

Clearly,

$$\begin{aligned} I_1 &= \int_0^{|x|/2} [\frac{\int_{B_t(x)} (1+|y|^2)^{-p\theta} dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &= c(1+|x|^2)^{-\frac{p\theta}{\gamma-1}} \int_0^{|x|/2} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \\ &= c(1+|x|^2)^{-\frac{p\theta}{\gamma-1}} |x|^{\frac{\beta\gamma}{\gamma-1}} = c(1+|x|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \end{aligned}$$

Take the slow rate $2\theta = \frac{\beta\gamma}{p-\gamma+1}$. Now, $\beta\gamma < 2p\theta < n$ in view of $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, and hence

$$\begin{aligned} I_2 &= c\int_{|x|/2}^{\infty} (\frac{\int_{B_t(x)} (1+|y|^2)^{-p\theta} dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &= c(x) \int_{|x|/2}^{\infty} (\frac{t^{n-2p\theta}}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} = c(x)(1+|x|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \end{aligned}$$

Thus, $I_1 + I_2 = c(x)u(x)$ for some double bounded $c(x)$.

Similarly, we also find a fast decaying solution. In fact, taking $2\theta = \frac{n-\beta\gamma}{\gamma-1}$, we also have $2p\theta > n$ from $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, and hence

$$\begin{aligned} I_2 &= \int_{|x|/2}^{\infty} (\frac{\int_{B_t(x)\cap B_1(0)} (1+|y|^2)^{-p\theta} dy + \int_{B_t(x)\setminus B_1(0)} (1+|y|^2)^{-p\theta} dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &= c(x) \int_{|x|/2}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} = c(x)(1+|x|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}. \end{aligned}$$

There also holds $I_1 + I_2 = c(x)(1+|x|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} = c(x)u(x)$.

Step 2. Nonexistence.

Substep 2.1. Let

$$0 < p < \frac{n(\gamma-1)}{n-\beta\gamma}. \tag{2.12}$$

Suppose that u solves (2.11), then

$$u(x) \geq c\int_{2|x|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} = \frac{c}{|x|^{a_0}}, \tag{2.13}$$

since $\int_{B_1(0)} u^p(y)dy \geq c$, where $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, we have

$$u(x) \geq c\int_{2|x|}^{\infty}(\frac{\int_{B_{t-|x|}(0)}|y|^{-pa_0}dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t} \geq c\int_{2|x|}^{\infty}(t^{\beta\gamma-pa_0})^{\frac{1}{\gamma-1}}\frac{dt}{t}. \tag{2.14}$$

When $\frac{p}{\gamma-1} \in (0, \frac{\beta\gamma}{n-\beta\gamma}]$, we have $\beta\gamma - pa_0 \geq 0$. Eq. (2.14) implies $u(x) = \infty$. It is impossible.

Next, we consider the case $\frac{p}{\gamma-1} \in (\frac{\beta\gamma}{n-\beta\gamma}, \frac{n}{n-\beta\gamma})$. Now (2.14) leads to

$$u(x) \geq \frac{c}{|x|^{a_1}},$$

where $a_1 = \frac{p}{\gamma-1}a_0 - \frac{\beta\gamma}{\gamma-1}$.

Write

$$a_j = \frac{p}{\gamma-1}a_{j-1} - \frac{\beta\gamma}{\gamma-1},\ j = 1, 2, \cdots. \tag{2.15}$$

Suppose that $a_k < a_{k-1}$ for $k = 1, 2, \cdots, j-1$. By virtue of (2.12), it follows

$$\begin{aligned} a_j - a_{j-1} &= (\tfrac{p}{\gamma-1} - 1)a_{j-1} - \tfrac{\beta\gamma}{\gamma-1} < (\tfrac{p}{\gamma-1} - 1)a_0 - \tfrac{\beta\gamma}{\gamma-1} \\ &= (\tfrac{p}{\gamma-1} - 1)\tfrac{n-\beta\gamma}{\gamma-1} - \tfrac{\beta\gamma}{\gamma-1} = (\tfrac{n-\beta\gamma}{(\gamma-1)^2})p - \tfrac{n}{\gamma-1} \\ &< (\tfrac{n-\beta\gamma}{(\gamma-1)^2})\tfrac{n(\gamma-1)}{n-\beta\gamma} - \tfrac{n}{\gamma-1} = 0. \end{aligned}$$

Thus, $\{a_j\}_{j=0}^{\infty}$ is decreasing as long as (2.12) is true.

Furthermore, we claim that there must be $j_0 > 0$ such that $a_{j_0} \leq 0$. This leads to $u(x) = \infty$, which contradicts with the fact that u is a positive solution.

In fact, by (2.15) we get

$$a_j = (\frac{p}{\gamma-1})^j a_0 - [1 + \frac{p}{\gamma-1} + \cdots + (\frac{p}{\gamma-1})^{j-1}]\frac{\beta\gamma}{\gamma-1}.$$

If $\frac{p}{\gamma-1} = 1$, then we can find a large j_0 such that

$$a_{j_0} = a_0 - j_0\frac{\beta\gamma}{\gamma-1} \leq 0.$$

If $\frac{p}{\gamma-1} \in (1, \frac{n}{n-\beta\gamma})$, then using $a_0 - \frac{\beta\gamma}{p-\gamma+1} < 0$ which is implied by (2.12), we can find a large j_0 such that

$$\begin{aligned} a_{j_0} &= (\frac{p}{\gamma-1})^{j_0} a_0 - \frac{(\frac{p}{\gamma-1})^{j_0} - 1}{\frac{p}{\gamma-1} - 1} \frac{\beta\gamma}{\gamma-1} \\ &= (\frac{p}{\gamma-1})^{j_0}(a_0 - \frac{\beta\gamma}{p-\gamma+1}) + \frac{\beta\gamma}{p-\gamma+1} \le 0. \end{aligned}$$

If $\frac{p}{\gamma-1} \in (0,1)$, letting $j \to \infty$, we get

$$a_j = (\frac{p}{\gamma-1})^j a_0 - \frac{1-(\frac{p}{\gamma-1})^j}{1-\frac{p}{\gamma-1}} \frac{\beta\gamma}{\gamma-1} \to \frac{\beta\gamma}{p-\gamma+1} < 0.$$

Thus, there must be j_0 such that $a_{j_0} \le 0$.

Substep 2.2. Let $p = \frac{n(\gamma-1)}{n-\beta\gamma}$. By the argument of Supstep 2.1, we can assume $p = \frac{n(\gamma-1)}{n-\beta\gamma} > 1$ here. We deduce the contradiction if u is a positive solution of (2.11).

For $R > 0$, denote $B_R(0)$ by B_R. By using the Hölder inequality, from (2.11) we deduce that for any $x \in B_R$,

$$\begin{aligned} &\int_0^R \int_{B_t(x)} u^p(y)dydt \\ &\le (\int_0^R (\int_{B_t(x)} u^p(y)dy)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}-1} dt)^{\gamma-1} (\int_0^R t^{\frac{n-\beta\gamma+\gamma-1}{2-\gamma}} dt)^{2-\gamma} \\ &= CR^{n-\beta\gamma+1} (\int_0^R (\frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t})^{\gamma-1}. \end{aligned}$$

Hence, exchanging the order of the integral variables, we have

$$\begin{aligned} u(x) &\ge c\int_0^R (\frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\ge cR^{-\frac{n-\beta\gamma+1}{\gamma-1}} (\int_0^R (\int_{B_t(x)} u^p(y)dy)dt)^{\frac{1}{\gamma-1}} \\ &\ge cR^{-\frac{n-\beta\gamma+1}{\gamma-1}} (\int_{B_R} u^p(y)(\int_{|x-y|}^R dt)dy)^{\frac{1}{\gamma-1}} \\ &\ge cR^{-\frac{n-\beta\gamma}{\gamma-1}} (\int_{B_{R/4}} u^p(y)dy)^{\frac{1}{\gamma-1}}. \end{aligned}$$

Therefore, we get

$$u^p(x) \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}} (\int_{B_{R/4}} u^p(y)dy)^{\frac{p}{\gamma-1}}. \tag{2.16}$$

Integrating on $B_{R/4}$ and using $p = \frac{n(\gamma-1)}{n-\beta\gamma}$ again, we get

$$\begin{aligned}
&\int_{B_{R/4}} u^p(x)dx \\
&\geq cR^{p\frac{\beta\gamma-n}{\gamma-1}} \int_{B_{R/4}} dx(\int_{B_{R/4}} u^p(y)dy)^{\frac{p}{\gamma-1}} \\
&\geq c(\int_{B_{R/4}} u^p(y)dy)^{\frac{p}{\gamma-1}}.
\end{aligned}$$

Here c is independent of R. Letting $R \to \infty$ and noting $p > \gamma - 1$, we have

$$\int_{R^n} u^p(x)dx < \infty. \tag{2.17}$$

Integrating (2.16) on $A_R = B_{R/4} \setminus B_{R/8}$ yields

$$\int_{A_R} u^p(x)dx \geq cR^{p\frac{\beta\gamma-n}{\gamma-1}} \int_{A_R} dx(\int_{B_{R/4}} u^p(y)dy)^{\frac{p}{\gamma-1}}.$$

By $p = \frac{n(\gamma-1)}{n-\beta\gamma}$, it follows

$$\int_{A_R} u^p(x)dx \geq c(\int_{B_{R/4}} u^p(y)dy)^{\frac{p}{\gamma-1}},$$

where c is independent of R. Letting $R \to \infty$, and noting (2.17), we obtain

$$\int_{R^n} u^p(y)dy = 0,$$

which implies $u \equiv 0$. It is impossible.

The proof is complete. $\square$

Remark 2.1. When $\beta = \alpha/2$ and $\gamma = 2$, (2.11) is reduced to (2.5). Theorem 2.5 is the generalization of Theorem 2.3.

2.3 γ-Laplace equation

Theorem 2.6. *(1) If $p > \frac{n(\gamma-1)}{n-\gamma}$, then the γ-Laplace equation*

$$-\Delta_\gamma u = c(x)u^p \tag{2.18}$$

has positive solutions for some double bounded $c(x)$.

(2) If $0 < p \leq \frac{n(\gamma-1)}{n-\gamma}$, then for any double bounded function $c(x)$, (2.18) has no any positive solution satisfying $\inf_{R^n} u = 0$.

Proof. (1) For $u(x) = \frac{1}{(1+|x|^m)^\theta}$ with $m = \frac{\gamma}{\gamma-1}$, similar to the derivation of (2.3), we have

$$\begin{aligned} -\Delta_\gamma u &= (1-\gamma)|U_r|^{\gamma-2}U_{rr} - \tfrac{n-1}{r}|U_r|^{\gamma-2}U_r \\ &= \tfrac{(m\theta)^{\gamma-1} r^{(m-1)(\gamma-2)+m-2}}{(1+r^m)^{(\theta+1)(\gamma-1)}}[\tfrac{-m(\theta+1)(\gamma-1)r^m}{1+r^m} + n - 1 + (\gamma-1)(m-1)] \\ &= \tfrac{(m\theta)^{\gamma-1}}{(1+r^m)^{(\theta+1)(\gamma-1)}}[\tfrac{n+[n-(\theta+1)\gamma]r^m}{1+r^m}]. \end{aligned} \tag{2.19}$$

Let $p > \frac{n(\gamma-1)}{n-\gamma}$. Take $m\theta = \frac{\gamma}{p-\gamma+1}$, then $n > (\theta+1)\gamma$. Therefore, (2.19) implies

$$-\Delta_\gamma u = c(r)u^{(\theta+1)(\gamma-1)/\theta} = c(r)u^p$$

for some double bounded $c(r)$. This result shows that (2.18) has a slowly decaying radial solution.

Moreover, if $p = \frac{n(\gamma-1)+\gamma}{n-\gamma}$, we can find another fast decaying solution with rate $m\theta = \frac{n-\gamma}{\gamma-1}$. Now, $n = (\theta+1)\gamma$ and hence (2.18) implies

$$-\Delta_\gamma u = c(r)u^{[(\theta+1)(\gamma-1)+1]/\theta} = c(r)u^p$$

for some double bounded $c(r)$.

(2) Suppose u solves (2.18) and satisfies $\inf_{R^n} u = 0$. According to Corollary 4.13 in [21], there exists $C > 0$ such that

$$\frac{1}{C}W_{1,\gamma}(cu^p)(x) \le u(x) \le CW_{1,\gamma}(cu^p)(x).$$

Since $c(x)$ is double bounded, we can see that

$$K(x) := \frac{u(x)}{W_{1,\gamma}(u^p)(x)}$$

is also double bounded. This shows that u solves

$$u(x) = K(x)W_{1,\gamma}(u^p)(x).$$

When $0 < p \le \frac{n(\gamma-1)}{n-\gamma}$, Theorem 2.5 shows that this Wolff type equation has no positive solution for any double bounded function $K(x)$. Therefore, we prove the nonexistence of positive solutions to (2.18) when $0 < p \le \frac{n(\gamma-1)}{n-\gamma}$. □

3 Systems with variable coefficients

3.1 HLS type system

Theorem 3.1. *There exist positive solutions u, v of the integral system involving the Riesz potentials*

$$\begin{cases} u(x) = c_1(x)\displaystyle\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} \\ v(x) = c_2(x)\displaystyle\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \end{cases} \tag{3.1}$$

for some double bounded functions $c_1(x)$ and $c_2(x)$, if and only if $pq > 1$ and

$$\max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\} < n - \alpha. \tag{3.2}$$

Proof. Step 1. Sufficiency.

Set

$$u(x) = \frac{1}{(1+|x|^2)^{\theta_1}}, \quad v(x) = \frac{1}{(1+|x|^2)^{\theta_2}}. \tag{3.3}$$

Similar to the argument in the proof of Theorem 2.3, we can find four pairs solutions.

(i) Take the slow rates

$$2\theta_1 = \frac{\alpha(q+1)}{pq-1}, \quad 2\theta_2 = \frac{\alpha(p+1)}{pq-1}.$$

Then $pq > 1$ as well as (3.2) lead to $\alpha < 2p\theta_1 < n$ and $\alpha < 2q\theta_2 < n$. Therefore,

$$\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c_1(x)}{(1+|x|^2)^{p\theta_1-\alpha/2}} = c_1(x)v(x),$$

$$\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} = \frac{c_2(x)}{(1+|x|^2)^{q\theta_2-\alpha/2}} = c_2(x)u(x),$$

for some double bounded functions $c_1(x)$ and $c_2(x)$. This consequence shows that (3.1) has a pair of radial solutions (u, v) as (3.3).

(ii) Moreover, if the stronger condition $p, q > \frac{n}{n-\alpha}$ holds, then we can find solutions u, v with the fast decay rate $2\theta_1 = 2\theta_2 = n - \alpha$. Now, $2p\theta_1 > n$ and $2q\theta_2 > n$, then

$$\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c_1(x)}{(1+|x|^2)^{(n-\alpha)/2}} = c_1(x)v(x),$$

$$\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} = \frac{c_2(x)}{(1+|x|^2)^{(n-\alpha)/2}} = c_2(x)u(x),$$

for some double bounded functions $c_1(x), c_2(x)$. Therefore, (3.1) has a pair of radial solutions (u, v) as (3.3).

(iii) If another stronger condition $pq > 1$ as well as

$$\frac{\alpha}{n-\alpha} < p < \frac{n}{n-\alpha}, \tag{3.4}$$

$$\frac{(q+1)\alpha}{pq-1} < n - \alpha, \tag{3.5}$$

holds, we can find a pair of solutions u, v. Now, u, v have two different fast decay rates.

We claim that if $pq > 1$, the condition (3.4) together with (3.5) are stronger than (3.2). In fact, we first see $p \leq q$. Otherwise, (3.4) implies $q < p < \frac{n}{n-\alpha}$,

which means $q[p(n-\alpha)-\alpha] < n$. This contradicts (3.5). From (3.5) and $p \le q$, it follows that $pq(n-\alpha)-n > q\alpha \ge p\alpha$. This leads to $\frac{(p+1)\alpha}{pq-1} < n-\alpha$. Combining this with (3.5) yields (3.2).

Take $2\theta_1 = n-\alpha$, $2\theta_2 = 2p\theta_1 - \alpha = p(n-\alpha)-\alpha$. Then (3.4) and (3.5) lead to $\alpha < 2p\theta_1 < n$ and $2q\theta_2 > n$. Therefore,

$$\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} = \frac{c_1(x)}{(1+|x|^2)^{p\theta_1-\alpha/2}} = c_1(x)v(x),$$

$$\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} = \frac{c_2(x)}{(1+|x|^2)^{(n-\alpha)/2}} = c_2(x)u(x),$$

for another double bounded functions $c_1(x), c_2(x)$. Therefore, (3.1) has a pair of radial solutions (u,v) as (3.3).

By the same argument above, we know that once $pq > 1$ as well as $\frac{\alpha}{n-\alpha} < q < \frac{n}{n-\alpha}$ and $\frac{(p+1)\alpha}{pq-1} < n-\alpha$, (3.1) has a pair of radial solutions (u,v) as (3.3). Now, u, v decay fast by two different rates.

(iv) We can find another pair of radial solutions to (3.1). They decay with fast rates which are different from (3.3). Now, we assume

$$u(x) = \frac{1}{(1+|x|^2)^{(n-\alpha)/2}}; \quad v(x) = \frac{\log|x|}{(1+|x|^2)^{(n-\alpha)/2}}.$$

It is easy to verify that u, v also solve (3.1) with some double bounded functions c_1, c_2.

Note. According to Corollary 1.3 (2) in [49], if $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ where p, q satisfy the critical condition $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{\alpha}{n}$, then u, v decay with only three rates as in (ii)-(iv).

Step 2. Necessity.

(i) If either $0 < pq \le 1$ or $pq > 1$ and $\max\{\frac{(p+1)\alpha}{pq-1}, \frac{(q+1)\alpha}{pq-1}\} > n-\alpha$, we prove the nonexistence.

Assume u, v are positive solutions of (3.1). First, for $|x| > R$,

$$u(x) \ge c\int_{B_R(0)} \frac{dy}{|x-y|^{n-\alpha}} \ge \frac{c}{|x|^{a_0}}.$$

Here $a_0 = n-\alpha$. By this estimate, for $|x| > R$ there holds

$$v(x) \ge c\int_{B_{|x|/2}(x)} \frac{|y|^{-pa_0}dy}{|x-y|^{n-\alpha}} \ge \frac{c}{|x|^{b_0}},$$

where $b_0 = pa_0 - \alpha$. This implies

$$u(x) \ge c\int_{B_{|x|/2}(x)} \frac{|y|^{-qb_0}dy}{|x-y|^{n-\alpha}} \ge \frac{c}{|x|^{a_1}},$$

for $|x| > R$, where $a_1 = qb_0 - \alpha$. By induction, we obtain that for $|x| > R$,

$$v(x) \ge \frac{c}{|x|^{b_k}}, \quad u(x) \ge \frac{c}{|x|^{a_k}}.$$

Here $a_0 = n - \alpha$, $b_k = pa_k - \alpha$ and $a_k = qb_{k-1} - \alpha$. Therefore, we have

$$\begin{aligned} a_j &= pqa_{j-1} - \alpha(q+1) = (pq)^2 a_{j-2} - \alpha(q+1)(1+pq) = \cdots \\ &= (pq)^j a_0 - \alpha(q+1)[1 + pq + \cdots + (pq)^{j-1}]. \end{aligned}$$

Case I: When $pq = 1$, for some large j_0, it follows

$$a_{j_0} = a_0 - \alpha(q+1)j_0 < 0.$$

Case II: When $0 < pq < 1$, letting $j \to \infty$, we get

$$a_j = (pq)^j a_0 - \alpha(q+1)\frac{1-(pq)^j}{1-pq} \to -\frac{\alpha(q+1)}{1-pq} < 0.$$

Therefore, we can find j_0 such that $a_{j_0} < 0$.

Case III: When $pq > 1$ and $\frac{\alpha(q+1)}{pq-1} > n - \alpha$.

Now, $a_0 < \frac{\alpha(q+1)}{pq-1}$. Thus, we deduce that for some large j_0,

$$a_{j_0} = (pq)^{j_0} a_0 - \alpha(q+1)\frac{(pq)^{j_0}-1}{pq-1} = (pq)^{j_0}[a_0 - \frac{\alpha(q+1)}{pq-1}] + \frac{\alpha(q+1)}{pq-1} < 0.$$

Case IV: When $pq > 1$ and $\frac{\alpha(p+1)}{pq-1} > n - \alpha$.

By an analogous argument of Case III, we can also find some k_0 such that $b_{k_0} < 0$.

These results imply $u(x) = \infty$ or $v(x) = \infty$. It is impossible. The contradiction shows the nonexistence of the positive solutions to (3.1).

(ii) If $pq > 1$ and $\max\{\frac{(p+1)\alpha}{pq-1}, \frac{(q+1)\alpha}{pq-1}\} = n - \alpha$, we prove the nonexistence.

The idea is the same as Step 3 in the proof of Theorem 2.19. Denote $B_R(0)$ by B. First,

$$u(x) \geq \frac{c}{(R+|x|)^{n-\alpha}}\int_B v^q(y)dy, \quad v(x) \geq \frac{c}{(R+|x|)^{n-\alpha}}\int_B u^p(y)dy.$$

Thus,

$$\begin{aligned} \int_B u^p(x)dx &\geq \frac{c}{R^{p(n-\alpha)-n}}(\int_B v^q(y)dy)^p, \\ \int_B v^q(x)dx &\geq \frac{c}{R^{q(n-\alpha)-n}}(\int_B u^p(y)dy)^q. \end{aligned}$$

Without loss of generality, assume $p \leq q$. Combining two results above with $\frac{(q+1)\alpha}{pq-1} = n - \alpha$ yields

$$\int_B v^q(x)dx \geq c(\int_B v^q(y)dy)^{pq},$$

where c is independent of R. Letting $R \to \infty$, we get $v \in L^q(R^n)$. On the other hand, we also obtain

$$\int_{A_R} v^q(x)dx \geq c(\int_B v^q(y)dy)^{pq}.$$

Letting $R \to \infty$ and noting $v \in L^q(R^n)$, we see $v \equiv 0$. It is impossible.

Theorem 3.1 is proved. $\square$

Corollary 3.2. *Let $k \in [1, n/2)$ be an integer and $pq > 1$. There exist positive solutions u, v of the semilinear Lane-Emden type system*

$$\begin{cases} (-\Delta)^k u(x) = c_1(x) v^q(x) \\ (-\Delta)^k v(x) = c_2(x) u^p(x) \end{cases} \tag{3.6}$$

for some double bounded functions $c_1(x)$ and $c_2(x)$, if and only if

$$\max\{\frac{2k(p+1)}{pq-1}, \frac{2k(q+1)}{pq-1}\} < n - 2k.$$

Proof. When $pq > 1$, Liu, Guo and Zhang [37] proved $(-\Delta)^i u > 0$ and $(-\Delta)^i v > 0$. Similar to the argument in [8] we can also establish the equivalence between (3.6) and (3.1). So Corollary 3.2 is a direct corollary of Theorem 3.1 with $\alpha = 2k$. $\square$

3.2 Wolff type system

Theorem 3.3. *There exist positive solutions u, v of the integral system involving the Wolff potentials*

$$\begin{cases} u(x) = c_1(x) W_{\beta,\gamma}(v^q)(x) \\ v(x) = c_2(x) W_{\beta,\gamma}(u^p)(x) \end{cases} \tag{3.7}$$

for some double bounded functions $c_1(x)$ and $c_2(x)$, if and only if $pq > (\gamma-1)^2$ and

$$\max\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\} < \frac{n-\beta\gamma}{\gamma-1}. \tag{3.8}$$

.

Proof. Step 1. Existence.

Insert (3.3) into $W_{\beta,\gamma}(u^p)$ and $W_{\beta,\gamma}(v^q)$. Similar to the argument in the proof of Theorem 2.5, we also discuss in four cases.

(i) Take the slow rates

$$2\theta_1 = \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \quad 2\theta_2 = \frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}.$$

Then, $pq > (\gamma-1)^2$ and (3.8) lead to $\beta\gamma < 2p\theta_1 < n$ and $\beta\gamma < 2q\theta_2 < n$. Therefore,

$$W_{\beta,\gamma}(u^p)(x) = \frac{c_1(x)}{(1+|x|^2)^{\frac{2p\theta_1-\beta\gamma}{2(\gamma-1)}}} = c_1(x)v(x),$$

$$W_{\beta,\gamma}(v^q)(x) = \frac{c_2(x)}{(1+|x|^2)^{\frac{2q\theta_2-\beta\gamma}{2(\gamma-1)}}} = c_2(x)u(x),$$

for some double bounded functions $c_1(x)$, $c_2(x)$. This implies that (3.7) has a pair of radial solutions (u, v) as (3.3).

(ii) If the stronger condition $p, q > \frac{n(\gamma-1)}{n-\beta\gamma}$ holds, we take the fast rate $2\theta_1 = 2\theta_2 = \frac{n-\beta\gamma}{\gamma-1}$. Then $2p\theta_1 > n$ and $2q\theta_2 > n$, and hence

$$W_{\beta,\gamma}(u^p)(x) = \frac{c_1(x)}{(1+|x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}} = c_1(x)v(x),$$

$$W_{\beta,\gamma}(v^q)(x) = \frac{c_2(x)}{(1+|x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}} = c_2(x)u(x),$$

for another double bounded functions $c_1(x), c_2(x)$. This implies that (3.7) has a pair of radial solutions (u, v) as (3.3) with fast decay rates.

(iii) Similar to the argument in Theorem 3.1, if $pq > (\gamma-1)^2$, the condition

$$\frac{\beta\gamma}{n-\beta\gamma} < p < \frac{n(\gamma-1)}{n-\beta\gamma}, \quad \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2} < \frac{n-\beta\gamma}{\gamma-1}$$

is also stronger than (3.8). When this stronger condition holds, then we take $2\theta_1 = \frac{n-\beta\gamma}{\gamma-1}$, $2\theta_2 = \frac{2p\theta_1-\beta\gamma}{\gamma-1} = p\frac{n-\beta\gamma}{(\gamma-1)^2} - \frac{\beta\gamma}{\gamma-1}$. Therefore, $\beta\gamma < 2p\theta_1 < n$ and $2q\theta_2 > n$, and hence

$$W_{\beta,\gamma}(u^p)(x) = \frac{c_1(x)}{(1+|x|^2)^{\frac{2p\theta_1-\beta\gamma}{2(\gamma-1)}}} = c_1(x)v(x),$$

$$W_{\beta,\gamma}(v^q)(x) = \frac{c_2(x)}{(1+|x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}} = c_2(x)u(x),$$

for another double bounded functions $c_1(x)$, $c_2(x)$. This shows (3.7) has radial solutions as (3.3).

Similar to the argument above, if another stronger condition $pq > (\gamma-1)^2$ as well as

$$\frac{\beta\gamma}{n-\beta\gamma} < q < \frac{n(\gamma-1)}{n-\beta\gamma}, \quad \frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2} < \frac{n-\beta\gamma}{\gamma-1}$$

holds, (3.7) also has radial solutions as (3.3) with two different fast rates $2\theta_2 = \frac{n-\beta\gamma}{\gamma-1}$, $2\theta_1 = q\frac{n-\beta\gamma}{(\gamma-1)^2} - \frac{\beta\gamma}{\gamma-1}$.

(iv) Eq. (3.7) also has another pair of radial solutions which also decay fast by two different rates. One decays with $\frac{n-\beta\gamma}{\gamma-1}$, and another decays with logarithmic order. Now, we assume

$$u(x) = \frac{1}{(1+|x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}}; \quad v(x) = \frac{(\log|x|)^{\frac{1}{\gamma-1}}}{(1+|x|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}}}.$$

It is easy to verify that u, v solve (3.7) with some double bounded functions c_1, c_2.

Step 2. Nonexistence.

Substep 2.1. Suppose either $0 < pq \le (\gamma-1)^2$ or

$$\max\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\} > \frac{n-\beta\gamma}{\gamma-1}.$$

Assume u, v are positive solutions of (3.7). Noting $\int_{B_R(0)} v^q(y)dy \ge c$, we obtain that for $|x| > R$,

$$u(x) \ge \int_{|x|+R}^{\infty} (\frac{\int_{B_R(0)} v^q(y)dy}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}} \frac{dt}{t} \ge c\int_{|x|+R}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \ge \frac{c}{|x|^{a_0}}.$$

Here $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, for $|x| > R$, there holds

$$v(x) \ge c\int_{2|x|}^{\infty} [\int_{B_{t-|x|}(0)\setminus B_{(t-|x|)/2}(0)} \frac{dy}{|y|^{pa_0}} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \ge c\int_{2|x|}^{\infty} t^{\frac{\beta\gamma-pa_0}{\gamma-1}} \frac{dt}{t}.$$

When $\beta\gamma - pa_0 \ge 0$, we see $v(x) = \infty$ for $|x| > R$. This implies the nonexistence of positive solutions of (3.7) since R is an arbitrary positive number. When $\beta\gamma - pa_0 < 0$, then

$$v(x) \ge \frac{c}{|x|^{b_0}}, \quad for\ |x| > R,$$

where $b_0 = \frac{pa_0-\beta\gamma}{\gamma-1}$. Similarly, using this estimate, we also obtain that if $\beta\gamma - qb_0 \ge 0$, then $u(x) = \infty$; if $\beta\gamma - qb_0 < 0$, then

$$u(x) \ge \frac{c}{|x|^{a_1}}, \quad for\ |x| > R,$$

where $a_1 = qb_0 - \beta\gamma$.

For $k = 1, 2, \cdots$, write

$$a_0 = \frac{n-\beta\gamma}{\gamma-1}, \quad b_k = \frac{pa_k-\beta\gamma}{\gamma-1}, \quad a_k = \frac{qb_{k-1}-\beta\gamma}{\gamma-1}.$$

By induction, we can obtain the following conclusions:

(i) If $a_k < 0$, then $u(x) = \infty$. This leads to the nonexistence. If $a_k \ge 0$, then $u(x) \ge \frac{c}{|x|^{a_k}}$ implies $v(x) \ge \frac{c}{|x|^{b_k}}$.

(ii) If $b_k < 0$, then $v(x) = \infty$. This also leads to the nonexistence. If $b_k \ge 0$, then $v(x) \ge \frac{c}{|x|^{b_k}}$ implies $u(x) \ge \frac{c}{|x|^{a_{k+1}}}$.

In view of

$$a_k = \frac{q}{\gamma-1}b_{k-1} - \frac{\beta\gamma}{\gamma-1} = \frac{pq}{(\gamma-1)^2}a_{k-1} - \frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1},$$

we deduce that

$$\begin{aligned} a_j &= \frac{pq}{(\gamma-1)^2}a_{j-1} - \frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1} \\ &= (\frac{pq}{(\gamma-1)^2})^2 a_{j-2} - \frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1}(1+\frac{pq}{(\gamma-1)^2}) = \cdots \\ &= (\frac{pq}{(\gamma-1)^2})^j a_0 - \frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1}[1+\frac{pq}{(\gamma-1)^2}+\cdots+(\frac{pq}{(\gamma-1)^2})^{j-1}]. \end{aligned}$$

When $\frac{pq}{(\gamma-1)^2}=1$, then for some large j_0,

$$a_{j_0}=a_0-\frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1}j_0<0.$$

This implies $u(x)=\infty$.

When $0<\frac{pq}{(\gamma-1)^2}<1$, letting $j\to\infty$, we get

$$a_j=(\frac{pq}{(\gamma-1)^2})^j a_0-\frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1}\frac{1-(\frac{pq}{(\gamma-1)^2})^j}{1-\frac{pq}{(\gamma-1)^2}}\to-\frac{\beta\gamma(q+\gamma-1)}{(\gamma-1)^2-pq}<0.$$

Therefore, we can find j_0 such that $a_{j_0}<0$. This implies $u(x)=\infty$.

When $\frac{pq}{(\gamma-1)^2}>1$ and $\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}>\frac{n-\beta\gamma}{\gamma-1}$, there holds $a_0<\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}$. We deduce that

$$\begin{aligned}a_{j_0}&=(\frac{pq}{(\gamma-1)^2})^{j_0}a_0-\frac{\beta\gamma}{\gamma-1}\frac{q+\gamma-1}{\gamma-1}\frac{(\frac{pq}{(\gamma-1)^2})^{j_0}-1}{\frac{pq}{(\gamma-1)^2}-1}\\&=(\frac{pq}{(\gamma-1)^2})^{j_0}[a_0-\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}]+\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}<0\end{aligned}$$

for some large j_0. We also see $u(x)=\infty$.

When $\frac{pq}{(\gamma-1)^2}>1$ and $\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}>\frac{n-\beta\gamma}{\gamma-1}$, there also holds $a_0<\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}$. By the same argument above, we handle b_k instead of a_k, we can also find some k_0 such that $b_{k_0}<0$. This implies $v(x)=\infty$.

Substep 2.2. Suppose $pq>(\gamma-1)^2$ and

$$\max\{\frac{\beta\gamma(p+\gamma-1)}{pq-(\gamma-1)^2},\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}\}=\frac{n-\beta\gamma}{\gamma-1}.$$

First, write $H:=\int_{B_t(x)}v^q(y)dy$. By the Hölder inequality,

$$\begin{aligned}\int_0^R Hdt&\le(\int_0^R H^{\frac{1}{\gamma-1}}t^{\frac{\beta\gamma-n}{\gamma-1}-1}dt)^{\gamma-1}(\int_0^R t^{\frac{n-\beta\gamma+\gamma-1}{2-\gamma}}dt)^{2-\gamma}\\&=CR^{n-\beta\gamma+1}(\int_0^R(\frac{H}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t})^{\gamma-1}.\end{aligned}$$

Therefore, exchanging the order of variables yields

$$\begin{aligned}u(x)\ge c\int_0^R(\frac{H}{t^{n-\beta\gamma}})^{\frac{1}{\gamma-1}}\frac{dt}{t}&\ge cR^{-\frac{n-\beta\gamma+1}{\gamma-1}}(\int_0^R Hdt)^{\frac{1}{\gamma-1}}\\&\ge cR^{-\frac{n-\beta\gamma}{\gamma-1}}(\int_{B_{R/4}}v^q(y)dy)^{\frac{1}{\gamma-1}}.\end{aligned}$$

Thus,

$$u^p(x)\ge cR^{-p\frac{n-\beta\gamma}{\gamma-1}}(\int_{B_{R/4}}v^q(y)dy)^{\frac{p}{\gamma-1}}. \tag{3.9}$$

Similarly,

$$v^q(x) \geq cR^{-q\frac{n-\beta\gamma}{\gamma-1}}(\int_{B_{R/4}} u^p(y)dy)^{\frac{q}{\gamma-1}}. \tag{3.10}$$

Without loss of generality, we suppose

$$\frac{\beta\gamma(q+\gamma-1)}{pq-(\gamma-1)^2} = \frac{n-\beta\gamma}{\gamma-1}. \tag{3.11}$$

Inserting (3.9) into (3.10) yields

$$v^q(x) \geq cR^{-q\frac{n-\beta\gamma}{\gamma-1}-pq\frac{n-\beta\gamma}{(\gamma-1)^2}+\frac{nq}{\gamma-1}}(\int_{B_{R/4}} v^q(y)dy)^{\frac{pq}{(\gamma-1)^2}}. \tag{3.12}$$

Integrating on $B_{R/4}$, we get

$$\int_{B_{R/4}} v^q(x)dx \geq cR^{-q\frac{n-\beta\gamma}{\gamma-1}(1+\frac{p}{\gamma-1})+n(\frac{q}{\gamma-1}+1)}(\int_{B_{R/4}} v^q(y)dy)^{\frac{pq}{(\gamma-1)^2}}. \tag{3.13}$$

We claim that the exponent of R is zero. In fact, $q\beta\gamma + n(\gamma-1) = \beta\gamma(q+\gamma-1) + (n-\beta\gamma)(\gamma-1)$. By (3.11), we obtain

$$q\beta\gamma + n(\gamma-1) = [pq-(\gamma-1)^2]\frac{n-\beta\gamma}{\gamma-1} + (\gamma-1)^2\frac{n-\beta\gamma}{\gamma-1} = pq\frac{n-\beta\gamma}{\gamma-1}.$$

Multiplying by $(\gamma-1)^{-1}$, we have

$$n(\frac{q}{\gamma-1}+1) = q\frac{n-\beta\gamma}{\gamma-1} + \frac{pq}{\gamma-1}\frac{n-\beta\gamma}{\gamma-1} = q\frac{n-\beta\gamma}{\gamma-1}(1+\frac{p}{\gamma-1}).$$

The claim is proved.

Letting $R \to \infty$ in (3.13), we see that $v \in L^q(R^n)$ in view of $pq > (\gamma-1)^2$.

Integrating (3.12) on $A_R := B_{R/4} \setminus B_{R/8}$ and letting $R \to \infty$, we also have $\int_{R^n} v^q(y)dy = 0$. It is impossible.

Thus, we complete our proof. □

3.3 γ-Laplace system

Theorem 3.4. *(1) If $pq > (\gamma-1)^2$ and*

$$\max\{\frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}\} < \frac{n-\gamma}{\gamma-1}, \tag{3.14}$$

then there exist positive solutions u, v of the γ-Laplace system

$$\begin{cases} -\Delta_\gamma u(x) = c_1(x)v^q(x), & x \in R^n, \\ -\Delta_\gamma v(x) = c_2(x)u^p(x), & x \in R^n \end{cases} \tag{3.15}$$

for some double bounded $c_1(x)$ and $c_2(x)$.

(2) For any double bounded functions $c_1(x)$ and $c_2(x)$, if one of the following conditions holds:

(i) $0 < pq \le (\gamma-1)^2$;

(ii) $pq > (\gamma-1)^2$ and

$$\max\{\frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}\} \ge \frac{n-\gamma}{\gamma-1}, \tag{3.16}$$

then (3.15) has no positive solutions u, v satisfying $\inf_{R^n} u = \inf_{R^n} v = 0$.

Proof. (1) *Existence.*

Let $m = \frac{\gamma}{\gamma-1}$. Take

$$u(x) = \frac{1}{(1+|x|^m)^{\theta_1}}, \quad v(x) = \frac{1}{(1+|x|^m)^{\theta_2}}.$$

Similar to the calculation in (2.19), we also obtain

$$\begin{aligned} -\Delta_\gamma u(x) &= \tfrac{(m\theta_1)^{\gamma-1}}{(1+r^m)^{(\theta_1+1)(\gamma-1)}}[\tfrac{n+[n-(\theta_1+1)\gamma]r^m}{1+r^m}], \\ -\Delta_\gamma v(x) &= \tfrac{(m\theta_2)^{\gamma-1}}{(1+r^m)^{(\theta_2+1)(\gamma-1)}}[\tfrac{n+[n-(\theta_2+1)\gamma]r^m}{1+r^m}]. \end{aligned}$$

Therefore, the signs of both sides of the results above show four cases.

(i) Take the slow decay rates

$$m\theta_1 = \frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \quad m\theta_2 = \frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}.$$

Then $pq > (\gamma-1)^2$ and (3.14) lead to

$$(\theta_1+1)\gamma < n, \quad \text{and} \quad (\theta_2+1)\gamma < n, \tag{3.17}$$

and hence

$$-\Delta_\gamma u(x) = \frac{c_1(r)}{(1+r^m)^{(\theta_1+1)(\gamma-1)}} = c_1(x)v^q(x),$$

$$-\Delta_\gamma v(x) = \frac{c_2(r)}{(1+r^m)^{(\theta_2+1)(\gamma-1)}} = c_2(x)u^p(x).$$

This shows that (3.15) has the radial solutions as (3.3) with slow decay rates.

(ii) Moreover, if $p = q = \frac{n(\gamma-1)+\gamma}{n-\gamma}$, then we take the fast decay rates $m\theta_1 = m\theta_2 = \frac{n-\gamma}{\gamma-1}$. This leads to $n = (\theta_1+1)\gamma = (\theta_2+1)\gamma$. Therefore,

$$-\Delta_\gamma u(x) = \frac{c_1(r)}{(1+r^m)^{(\theta_1+1)(\gamma-1)+1}} = c_1(x)v^q(x),$$

$$-\Delta_\gamma v(x) = \frac{c_2(r)}{(1+r^m)^{(\theta_2+1)(\gamma-1)+1}} = c_2(x)u^p(x).$$

This shows that (3.15) has the radial solutions as (3.3) with fast decay rates.

(iii) If $\frac{\gamma(q+\gamma)}{pq-(\gamma-1)^2} = \frac{n-\gamma}{\gamma-1}$, then we take other fast decay rates $m\theta_1 = \frac{n-\gamma}{\gamma-1}$, $m\theta_2 = p\frac{n-\gamma}{(\gamma-1)^2} - \frac{\gamma}{\gamma-1}$. Thus, $n = (\theta_1+1)\gamma$, $n > (\theta_2+1)\gamma$. Therefore,

$$-\Delta_\gamma u(x) = \frac{c_1(r)}{(1+r^m)^{(\theta_1+1)(\gamma-1)+1}} = c_1(x)v^q(x),$$

$$-\Delta_\gamma v(x) = \frac{c_2(r)}{(1+r^m)^{(\theta_2+1)(\gamma-1)}} = c_2(x)u^p(x).$$

This shows that (3.15) has the radial solutions as (3.3) with the second fast decay rates.

Similar to the argument above, if $\frac{\gamma(p+\gamma)}{pq-(\gamma-1)^2} = \frac{n-\gamma}{\gamma-1}$ holds, (3.7) also has radial solutions as (3.3) with the third fast rates $m\theta_2 = \frac{n-\gamma}{\gamma-1}$, $m\theta_1 = q\frac{n-\gamma}{(\gamma-1)^2} - \frac{\gamma}{\gamma-1}$.

(iv) Eq. (3.7) also has another pair of radial solutions which also decay fast with the different rates. One decays with $\frac{n-\gamma}{\gamma-1}$, and another decays with logarithmic order. Now, we assume

$$u(x) = \frac{1}{(1+|x|^m)^{\frac{n-\gamma}{\gamma}}}; \quad v(x) = \frac{(\log|x|)^{\frac{1}{\gamma-1}}}{(1+|x|^m)^{\frac{n-\gamma}{\gamma}}}.$$

It is easy to verify that u, v solve (3.7) with some double bounded functions c_1, c_2.

(2) *Nonexistence.*

Suppose u, v are positive solutions of (3.15) satisfying $\inf_{R^n} u = \inf_{R^n} v = 0$. According to Corollary 4.13 in [21], there exists $C > 0$ such that

$$\frac{1}{C}W_{1,\gamma}(c_1 v^q)(x) \le u(x) \le CW_{1,\gamma}(c_1 v^q)(x),$$

$$\frac{1}{C}W_{1,\gamma}(c_2 u^p)(x) \le v(x) \le CW_{1,\gamma}(c_2 u^p)(x).$$

Since c_1 and c_2 are double bounded, we can find two other double bounded functions $K_1(x)$ and $K_2(x)$ such that

$$u(x) = K_1(x)W_{1,\gamma}(v^q)(x), \quad v(x) = K_2(x)W_{1,\gamma}(u^p)(x).$$

By Theorem 3.3 with $\beta = 1$, we can see the nonexistence. $\square$

4 Finite energy solutions: scalar equations

In this section, we consider the critical conditions associated with the existence of the positive solutions when the coefficient $c(x) \equiv Constant$. Without loss of generality, we take $c(x) \equiv 1$.

4.1 Critical exponents and scaling invariants

Take a scaling transform $u_\mu(x) = \mu^{\frac{n-2}{2}} u(\mu x)$. Assume u solves $-\Delta u = u^{\frac{n+2}{n-2}}$. By a simply calculation, we have

$$-\Delta u_\mu = u_\mu^{\frac{n+2}{n-2}} \quad \textit{and} \quad \|u\|_{\frac{2n}{n-2}} = \|u_\mu\|_{\frac{2n}{n-2}}.$$

For the higher order equation, the corresponding result above is still true.

Furthermore, we have the more general result.

Theorem 4.1. *The HLS type eauation*

$$u(x) = \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \tag{4.1}$$

and the energy $\|u\|_{L^{p+1}(R^n)}$ *are invariant under the scaling transform, if and only if*

$$p = \frac{n+\alpha}{n-\alpha}. \tag{4.2}$$

Proof. Take the scaling transform

$$u_\mu(x) = \mu^\sigma u(\mu x).$$

Then

$$\begin{aligned} u_\mu(x) &= \mu^\sigma \int_{R^n} \frac{u^p(y)dy}{|\mu x - y|^{n-\alpha}} = \mu^\sigma \int_{R^n} \frac{\mu^n u^p(\mu z)dz}{|\mu(x-z)|^{n-\alpha}} \\ &= \mu^\sigma \int_{R^n} \frac{\mu^{n-p\sigma} u_\mu^p(z)dz}{\mu^{n-\alpha}|x-z|^{n-\alpha}} = \mu^{\sigma - p\sigma + \alpha} \int_{R^n} \frac{u_\mu^p(y)dy}{|x-y|^{n-\alpha}}. \end{aligned}$$

If u_μ still solves (4.1), then

$$\sigma = \frac{\alpha}{p-1}. \tag{4.3}$$

Next,

$$\int_{R^n} u_\mu^{p+1}(x)dx = \int_{R^n} [\mu^\sigma u(\mu x)]^{p+1} dx = \mu^{\sigma(p+1)-n} \int_{R^n} u^{p+1}(z)dz.$$

If the $L^{p+1}(R^n)$-norm is invariant, then there holds

$$\sigma = \frac{n}{p+1}.$$

Combining this with (4.3), we get (4.2).

On the contrary, if (4.2) is true, then we can also deduce the invariance by the same calculation above. □

Theorem 4.2. *The Wolff type equation*

$$u(x) = \int_0^\infty \left(\frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \tag{4.4}$$

and the energy $\|u\|_{L^{p+\gamma-1}(R^n)}$ are invariant under the scaling transform, if and only if

$$p = \frac{n+\beta\gamma}{n-\beta\gamma}(\gamma-1). \tag{4.5}$$

In addition, (4.4) and another energy $\|u\|_{L^{p+1}(R^n)}$ are invariant under the scaling transform, if and only if

$$p = \gamma^* - 1 \quad (\textit{where} \quad \gamma^* = \frac{n\gamma}{n-\beta\gamma}). \tag{4.6}$$

Proof. Take the scaling transform

$$u_\mu(x) = \mu^\sigma u(\mu x).$$

Then

$$\begin{aligned} u_\mu(x) \quad &= \mu^\sigma \int_0^\infty \left(\frac{\int_{B_t(\mu x)} u^p(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &= \mu^\sigma \int_0^\infty \left(\frac{\int_{B_t(\mu x)} u^p(\mu z)d(\mu z)}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &= \mu^\sigma \int_0^\infty \left(\frac{\int_{B_s(x)} \mu^{n-p\sigma} u_\mu^p(z)dz}{(\mu s)^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{ds}{s} \\ &= \mu^{\sigma+\frac{\beta\gamma-p\sigma}{\gamma-1}} \int_0^\infty \left(\frac{\int_{B_s(x)} u_\mu^p(z)dz}{s^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{ds}{s}. \end{aligned}$$

Thus, u_μ solves (4.4) if and only if

$$\sigma = \frac{\beta\gamma}{p-\gamma+1}. \tag{4.7}$$

Next,

$$\int_{R^n} u_\mu^{p+\gamma-1}(x)dx = \int_{R^n} [\mu^\sigma u(\mu x)]^{p+\gamma-1}dx = \mu^{\sigma(p+\gamma-1)-n} \int_{R^n} u^{p+\gamma-1}(z)dz.$$

The $L^{p+\gamma-1}(R^n)$-norm is invariant, if and only if

$$\sigma = \frac{n}{p+\gamma-1}.$$

Combining this with (4.7), we get (4.5).

At last,

$$\int_{R^n} u_\mu^{p+1}(x)dx = \int_{R^n} [\mu^\sigma u(\mu x)]^{p+1}dx = \mu^{\sigma(p+1)-n} \int_{R^n} u^{p+1}(z)dz.$$

The $L^{p+1}(R^n)$-norm is invariant, if and only if

$$\sigma = \frac{n}{p+1}.$$

Combining this with (4.7), we get (4.6). $\square$

Since the corresponding result of the γ-Laplace equation can not be covered by that of the Wolff type equation, we should point out the following conclusion.

Theorem 4.3. *The γ-Laplace equation*

$$-\Delta_\gamma u(x) = u^p(x) \tag{4.8}$$

and the energy $\|u\|_{L^{p+\gamma-1}(R^n)}$ are invariant under the scaling transform, if and only if (4.5) with $\beta = 1$ holds. In addition, (4.8) and another energy $\|u\|_{L^{p+1}(R^n)}$ are invariant under the scaling transform, if and only if (4.6) with $\beta = 1$ holds.

Proof. Suppose u_μ is a solution of (4.8). Then

$$-div(|\nabla u_\mu|^{\gamma-2}\nabla u_\mu) = u_\mu^p.$$

$$-\mu^{\sigma(\gamma-1)} div_x(|\nabla_x u(\mu x)|^{\gamma-2}\nabla_x u(\mu x)) = \mu^{p\sigma} u^p(\mu x).$$

Let $y = \mu x$, then

$$-\mu^{\sigma(\gamma-1)+\gamma} div_y(|\nabla_y u(y)|^{\gamma-2}\nabla_y u(y)) = \mu^{p\sigma} u^p(y).$$

This result shows that the equation is invariant if and only if

$$\sigma = \frac{\gamma}{p-\gamma+1}.$$

By the same argument as in Theorem 4.2, the invariance of the energy is equivalent to

$$\sigma = \frac{n}{p+\gamma-1}.$$

Eliminating σ from the two formulas above yields $p = \frac{n+\gamma}{n-\gamma}(\gamma-1)$.

The proof that (4.6) with $\beta = 1$ is the sufficient and necessary condition is the same as the argument above. $\square$

4.2 HLS type equation

Theorem 4.4. *Assume $u > 0$ is a classical solution of*

$$-\Delta u(x) = u^p(x), \quad x \in R^n. \tag{4.9}$$

Assume $u \in L^{2^*}(R^n)$. Then $\nabla u \in L^2(R^n)$ if and only if $u \in L^{p+1}(R^n)$.

A classical positive solution $u \in L^{2^*}(R^n)$ of (4.9) is called *finite energy solution*, if $u \in L^{p+1}(R^n)$ or $\nabla u \in L^2(R^n)$.

Proof. Take smooth function $\zeta(x)$ satisfying

$$\begin{array}{ll} \zeta(x) = 1, & for\ |x| \leq 1; \\ \zeta(x) \in [0,1], & for\ |x| \in [1,2]; \\ \zeta(x) = 0, & for\ |x| \geq 2. \end{array}$$

Define the cut-off function

$$\zeta_R(x) = \zeta(\frac{x}{R}). \tag{4.10}$$

Multiplying (4.9) by $u\zeta_R^2$ and integrating on $D := B_{3R}(0)$, we have

$$-\int_D u\zeta_R^2 \Delta u dx = \int_D u^{p+1}\zeta_R^2 dx.$$

Integrating by parts, we obtain

$$\int_D |\nabla u|^2 \zeta_R^2 dx + 2\int_D u\zeta_R \nabla u \nabla \zeta_R dx = \int_D u^{p+1}\zeta_R^2 dx. \tag{4.11}$$

Applying the Young inequality, we get

$$|\int_D u\zeta_R \nabla u \nabla \zeta_R dx| \leq \delta \int_D |\nabla u|^2 \zeta_R^2 dx + C\int_D u^2 |\nabla \zeta_R|^2 dx \tag{4.12}$$

for any $\delta \in (0, 1/2)$. If $u \in L^{2^*}(R^n)$, we can find $C >$ which is independent of R such that

$$\int_D u^2 |\nabla \zeta_R|^2 dx \leq C. \tag{4.13}$$

If $u \in L^{p+1}(R^n) \cap L^{2^*}(R^n)$, then (4.11)-(4.13) imply $\int_D |\nabla u|^2 \zeta_R^2 dx \leq C$. Letting $R \to \infty$ yields

$$\nabla u \in L^2(R^n).$$

This and $u \in L^{2^*}(R^n)$ show that for some $R = R_j \to \infty$,

$$R\int_{\partial D} (|\nabla u|^2 + u^{2^*}) ds \to 0,$$

by Proposition 2.1. Therefore,

$$|\int_{\partial D} u\partial_\nu u ds| \leq (\int_{\partial D} u^{2^*} ds)^{1/2^*} (\int_{\partial D} |\nabla u|^2 ds)^{1/2} R^{(n-1)(1/2-1/2^*)} \to 0, \tag{4.14}$$

when $R \to \infty$. Multiplying (4.9) by u yields

$$\int_D u^{p+1} dx = \int_D |\nabla u|^2 dx - \int_{\partial D} u\partial_\nu u ds. \tag{4.15}$$

Letting $R \to \infty$ and using the result above, we have $\|\nabla u\|_2^2 = \|u\|_{p+1}^{p+1}$.

If $\nabla u \in L^2(R^n)$ and $u \in L^{2^*}(R^n)$, (4.14) still holds. If letting $R \to \infty$ in (4.15) and inserting (4.14) into it, we obtain $\|u\|_{p+1}^{p+1} = \|\nabla u\|_2^2$ and hence $u \in L^{p+1}(R^n)$. $\square$

Next, we use the Pohozaev type identity in integral forms to discuss the existence of the finite energy solutions of (4.1). A positive classical solution u of (4.1) is called *finite energy solution*, if $u \in L^{p+1}(R^n)$.

Theorem 4.5. *The HLS type integral equation (4.1) has positive classical solution in $L^{p+1}(R^n)$ if and only if (4.2) holds.*

Proof. If (4.2) holds, (4.1) exists a unique class of finite energy solutions (cf. [11] or [35]):

$$u(x) = c\left(\frac{t}{t^2 + |x - x_0|^2}\right)^{(n-\alpha)/2}.$$

Here c, t are positive constants.

On the contrary, if $u \in L^{p+1}(R^n)$ solves (4.1), we claim that (4.2) is true. In fact, for any $\mu \neq 0$, from (4.1) it follows

$$u(\mu x) = \int_{R^n} \frac{u^p(y)dy}{|\mu x - y|^{n-\alpha}} = \int_{R^n} \frac{\mu^n u^p(\mu z)dz}{|\mu(x - z)|^{n-\alpha}} = \mu^\alpha \int_{R^n} \frac{u^p(\mu z)dz}{|x - z|^{n-\alpha}}.$$

Differentiate both sides with respect to μ. Then,

$$x \cdot \nabla u(\mu x) = \alpha \mu^{\alpha-1} \int_{R^n} \frac{u^p(\mu z)dz}{|x - z|^{n-\alpha}} + \mu^\alpha \int_{R^n} \frac{pu^{p-1}(\mu z)(z \cdot \nabla u)dz}{|x - z|^{n-\alpha}}.$$

Letting $\mu = 1$ yields

$$x \cdot \nabla u(x) = \alpha u(x) + \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x - z|^{n-\alpha}}. \tag{4.16}$$

To handle the last term of the right hand side of (4.16), we integrate by parts to get

$$\int_{B_R} \frac{z \cdot \nabla u^p(z)dz}{|x - z|^{n-\alpha}} = R \int_{\partial B_R} \frac{u^p(z)ds}{|x - z|^{n-\alpha}} - I_R(x) \tag{4.17}$$

for any $R > 0$. Here $B_R = B_R(0)$ and

$$I_R(x) = n \int_{B_R} \frac{u^p(z)dz}{|x - z|^{n-\alpha}} + (n - \alpha) \int_{B_R} \frac{(z \cdot (x - z))u^p(z)}{|x - z|^{n-\alpha+2}} dz.$$

Next, we claim the first term of the right hand side of (4.17) converges to zero as $R \to \infty$. In fact, for suitably large R,

$$\begin{aligned} & R \int_{\partial B_R} \frac{u^p(z)ds}{|x - z|^{n-\alpha}} \\ \leq & \, CR^{1+\alpha-n} (R \int_{\partial B_R} u^{p+1} ds)^{\frac{p}{p+1}} R^{-\frac{p}{p+1}} R^{\frac{n-1}{p+1}} \\ = & \, CR^{\alpha-n+\frac{n}{p+1}} (R \int_{\partial B_R} u^{p+1} ds)^{\frac{p}{p+1}}. \end{aligned} \tag{4.18}$$

By Theorem 2.3, we see that $p \geq \frac{n}{n-\alpha}$. So $\alpha - n + \frac{n}{p+1} < 0$. In addition, using Proposition 2.1 and $u \in L^{p+1}(R^n)$, we can find $R_j \to \infty$ such that

$$R_j \int_{\partial B_{R_j}} u^{p+1} ds \to 0. \tag{4.19}$$

Let $R = R_j \to \infty$ in (4.18), we verify our claim.

Multiplying (4.16) by $u^p(x)$ and applying the claim above, we obtain

$$\begin{aligned}
&\int_{R^n} u^p(x)(x \cdot \nabla u(x))dx \\
&= \alpha \int_{R^n} u^{p+1}(x)dx + \int_{R^n} u^p(x)dx \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x-z|^{n-\alpha}} \\
&= \alpha \int_{R^n} u^{p+1}(x)dx - n \int_{R^n} u^p(x)dx \int_{R^n} \frac{u^p(z)dz}{|x-z|^{n-\alpha}} \\
&\quad -(n-\alpha) \int_{R^n} \int_{R^n} \frac{(z \cdot (x-z))u^p(x)u^p(z)}{|x-z|^{n-\alpha+2}} dzdx.
\end{aligned}$$

By virtue of $z \cdot (x-z) + x \cdot (z-x) = -|x-z|^2$, it follows that

$$\begin{aligned}
&\int_{R^n} u^p(x)(x \cdot \nabla u(x))dx \\
&= \alpha \int_{R^n} u^{p+1}(x)dx - n \int_{R^n} u^{p+1}(x)dx \\
&\quad + \frac{n-\alpha}{2} \int_{R^n} \int_{R^n} \frac{u^p(x)u^p(z)}{|x-z|^{n-\alpha}} dzdx \\
&= -\frac{n-\alpha}{2} \int_{R^n} u^{p+1}(x)dx.
\end{aligned}$$

On the other hand, integrating by parts an using (4.19), we get

$$\int_{R^n} u^p(x)(x \cdot \nabla u(x))dx = \frac{1}{p+1} \int_{R^n} (x \cdot \nabla u^{p+1}(x))dx = \frac{-n}{p+1} \int_{R^n} u^{p+1}(x)dx.$$

Combining this with the result above, we deduce that

$$\frac{1}{p+1} = \frac{n-\alpha}{2n}.$$

This is (4.2). Theorem 4.5 is proved. $\square$

Corollary 4.6. *Let $k \in [1, n/2)$ be an integer and $p > 1$. The $2k$-order Lane-Emden PDE*

$$(-\Delta)^k u(x) = u^p(x), \quad u > 0 \text{ in } R^n, \tag{4.20}$$

has positive classical solution in $L^{p+1}(R^n)$ if and only if $p = \frac{n+2k}{n-2k}$.

Proof. When $p > 1$, Corollary 2.4 shows that (4.20) is equivalent to the HLS type equation (4.1) with $\alpha = 2k$. According to Theorem 4.5, we have the corresponding critical conditions $p = \frac{n+2k}{n-2k}$ for the existence of the finite energy solutions of the (4.9). $\square$

Remark 4.1. Theorem 4.5 shows another critical condition (4.2) for the existence of the positive solutions to (4.1). Since the finite energy solutions class of (4.1) is smaller than the positive solutions class of (2.5), the critical condition (4.2) is stronger than (2.6).

4.3 γ-Laplace equation

Serrin and Zou [47] proved that γ-Laplace equation has positive classical solutions if and only if $p \geq \gamma^* - 1$, where $\gamma^* = \frac{n\gamma}{n-\gamma}$. Naturally, we conjecture that γ-Laplace equation has the finite energy solution if and only if $p = \gamma^* - 1$.

To define the finite energy solution, we first introduce the following theorem. It is a natural generalization of Theorem 4.4.

Theorem 4.7. *Assume $u > 0$ is a classical solution of the γ-Laplace equation (4.8). Assume $u \in L^{\gamma^*}(R^n)$ with $\gamma^* = \frac{n\gamma}{n-\gamma}$. Then $\nabla u \in L^\gamma(R^n)$ if and only if $u \in L^{p+1}(R^n)$. In addition, $\|\nabla u\|_\gamma^\gamma = \|u\|_{p+1}^{p+1}$.*

A classical positive solution $u \in L^{\gamma^*}(R^n)$ of (4.8) is called *finite energy solution* if $u \in L^{p+1}(R^n)$ or $\nabla u \in L^\gamma(R^n)$.

Proof. Let $u \in L^{\gamma^*}(R^n)$. Take a cut-off function ζ_R as (4.10). Using the Hölder inequality, we get

$$\int_D u^\gamma |\nabla \zeta_R|^\gamma dx \leq \|u\|_{\gamma^*,D}^\gamma \|\nabla \zeta\|_{\frac{\gamma\gamma^*}{\gamma^*-\gamma},D}^\gamma \leq C, \tag{4.21}$$

where $D = B_{2R}(0)$, and $C > 0$ is independent of R.

(1) Sufficiency. Supposing $u \in L^{p+1}(R^n) \cap L^{\gamma^*}(R^n)$ solves (4.8), we claim $\nabla u \in L^\gamma(R^n)$ and $\|\nabla u\|_\gamma^\gamma = \|u\|_{p+1}^{p+1}$.

Multiplying (4.8) by $u\zeta_R^\gamma$ and integrating by parts on D, we obtain

$$\int_D |\nabla u|^\gamma \zeta_R^\gamma dx + \gamma \int_D |\nabla u|^{\gamma-2}(u\zeta_R^{\gamma-1})\nabla u \nabla \zeta_R dx = \int_D u^{p+1}\zeta_R^\gamma dx. \tag{4.22}$$

Using the Young inequality, from (4.22) we deduce that for any $\delta \in (0, 1/2)$,

$$\begin{aligned}\int_D |\nabla u|^\gamma \zeta_R^\gamma dx &\leq C|\int_D |\nabla u|^{\gamma-2}(u\zeta_R^{\gamma-1})\nabla u \nabla \zeta_R dx| + \int_D u^{p+1}\zeta_R^\gamma dx \\ &\leq \delta \int_D |\nabla u|^\gamma \zeta_R^\gamma dx + C\int_D u^\gamma |\nabla \zeta_R|^\gamma dx + \int_D u^{p+1}\zeta_R^\gamma dx.\end{aligned}$$

Combining this result with (4.21), we see that $\int_D |\nabla u|^\gamma \zeta_R^\gamma dx \leq C$. Let $R \to \infty$, then

$$\nabla u \in L^\gamma(R^n).$$

From this result as well as $u \in L^{\gamma^*}(R^n)$, we use Proposition 2.1 to deduce that for some R_j (denoted by R),

$$R\int_{\partial D}(|\nabla u|^\gamma + u^{\gamma^*})ds < o(1), \quad as\ R \to \infty. \tag{4.23}$$

Multiplying (4.8) by u and integrating on D, we have

$$\int_D u^{p+1}dx = -\int_D u\Delta_\gamma u dx = \int_D |\nabla u|^\gamma dx - \int_{\partial D} u|\nabla u|^{\gamma-2}\partial_\nu u ds. \tag{4.24}$$

By means of the Hölder inequality and (4.23), we get

$$\begin{aligned}
&|\int_{\partial D} u|\nabla u|^{\gamma-2}\partial_\nu u ds| \\
&\leq [R\int_{\partial D} |\nabla u|^\gamma ds]^{\frac{\gamma-1}{\gamma}} R^{\frac{1-\gamma}{\gamma}} [R\int_{\partial D} u^{\gamma^*} ds]^{1/\gamma^*} R^{-1/\gamma^*} R^{(n-1)(1/\gamma-1/\gamma^*)} \\
&< o(1).
\end{aligned}$$

Letting $R \to \infty$ in (4.24), we have

$$\int_{R^n} |\nabla u|^\gamma dx = \int_{R^n} u^{p+1} dx. \tag{4.25}$$

(2) Necessity. Let $u > 0$ solve (4.8). If $\nabla u \in L^\gamma(R^n)$ and $u \in L^{\gamma^*}(R^n)$, then (4.23) is still true. Using (4.23) to handle the last term of the right hand side of (4.24), we also derive (4.25) and hence $u \in L^{p+1}(R^n)$.

□

Theorem 4.8. *The γ-Laplace equation (4.8) has a classical solution satisfying $\nabla u \in L^\gamma(R^n)$ if and only if*

$$p = \gamma^* - 1. \tag{4.26}$$

Proof. If $p = \gamma^* - 1$, according to p.328 in [13], (4.8) admits a class of solutions

$$u(x) = \frac{d}{[1 + D(d^{\frac{\gamma}{n-\gamma}}|x|^{\frac{\gamma}{\gamma-1}})]^{\frac{n-\gamma}{\gamma}}}.$$

Here d, D are positive constants.

Next, we prove the sufficiency. Write $B = B_R(0)$. Multiplying the equation with $(x \cdot \nabla u)$ and integrating on B, we obtain

$$\begin{aligned}
&\int_B |\nabla u|^{\gamma-2}\nabla u \nabla(x\cdot\nabla u)dx - \int_{\partial B} |\nabla u|^{\gamma-2}(\nu\cdot\nabla u)(x\cdot\nabla u)ds \\
&= \int_B u^p(x\cdot\nabla u)dx.
\end{aligned}$$

Here ν is the unit outward normal vector to ∂B. Noting

$$\nabla u\nabla(x\cdot\nabla u) = |\nabla u|^2 + \frac{1}{2}x\cdot\nabla(|\nabla u|^2)$$

and $x = |x|\nu$, we have

$$\begin{aligned}
&\int_B |\nabla u|^\gamma dx + \frac{1}{\gamma}\int_B x\cdot\nabla(|\nabla u|^\gamma)dx - R\int_{\partial B} |\nabla u|^{\gamma-2}|\partial_\nu u|^2 ds \\
&= \frac{1}{p+1}\int_B x\cdot\nabla u^{p+1}dx.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}&(1-\frac{n}{\gamma})\int_B|\nabla u|^\gamma dx+\frac{R}{\gamma}\int_{\partial B}|\nabla u|^\gamma ds-R\int_{\partial B}|\nabla u|^{\gamma-2}|\partial_\nu u|^2ds\\&=\frac{R}{p+1}\int_{\partial B}u^{p+1}ds-\frac{n}{p+1}\int_B u^{p+1}dx.\end{aligned}\tag{4.27}$$

According to Theorem 4.7, $\nabla u\in L^\gamma(R^n)$ implies $u\in L^{\gamma^*}(R^n)\cap L^{p+1}(R^n)$. Therefore, by Proposition 2.1, we can find $R_j\to\infty$, such that

$$R_j\int_{\partial B_{R_j}}(u^{p+1}+|\nabla u|^\gamma)ds\to 0.$$

Let $R=R_j\to\infty$ in (4.27). By means of the result above, we deduce that

$$(1-\frac{n}{\gamma})\int_{R^n}|\nabla u|^\gamma dx=-\frac{n}{p+1}\int_{R^n}u^{p+1}dx.$$

Inserting (4.25) into this result yields $p=\gamma^*-1$. □

Remark 4.2.

1. For the Wolff type equation (4.4), we do not know whether (4.5) is the necessary and sufficient condition for the existence of positive solution in $L^{p+\gamma-1}(R^n)$.

2. A surprising observation is, when $\gamma\neq 2$, the critical condition (4.26) is different from (4.5) with $\beta=1$. One reason is that the solution of (4.8) only solves a Wolff type equation with variable coefficient instead of (4.4). Another reason is that the finite energy functions spaces $L^{p+1}(R^n)$ and $L^{p+\gamma-1}(R^n)$ are also different except for $\gamma=2$. This distinction shows that (4.2) and (4.26) are not the same class critical exponents. For γ-Laplace equation, besides the divided number in Theorem 2.6, we also have two critical exponents mentioned above. The relation of them is

$$\frac{n(\gamma-1)}{n-\gamma}<\frac{n+\gamma}{n-\gamma}(\gamma-1)<\gamma^*-1$$

as long as $\gamma\in(1,2)$. This is also led to by the difference of the existence spaces of positive solutions.

5 Finite energy solutions: system

5.1 Critical conditions and scaling invariants

Theorem 5.1. *(1) Both the semilinear Lane-Emden type system*

$$\begin{cases}-\Delta u=v^q\\-\Delta v=u^p.\end{cases}\tag{5.1}$$

and the energy integrals $\|u\|_{L^{p+1}(R^n)}$ *and* $\|v\|_{L^{q+1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-2}{n}. \tag{5.2}$$

(2) Both the γ*-Laplace system*

$$\begin{cases} -\Delta_\gamma u = v^q, \\ -\Delta_\gamma v = u^p. \end{cases} \tag{5.3}$$

and the energy integrals $\|u\|_{L^{p+\gamma-1}(R^n)}$ *and* $\|v\|_{L^{q+\gamma-1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$\frac{1}{p+\gamma-1}+\frac{1}{q+\gamma-1}=\frac{n-\gamma}{n(\gamma-1)}. \tag{5.4}$$

In addition, (5.3) and the energy integrals $\|u\|_{L^{p+1}(R^n)}$ *and* $\|v\|_{L^{q+1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$p=q \quad or \quad \gamma=2. \tag{5.5}$$

(3) The HLS type system

$$\begin{cases} u(x)=\displaystyle\int_{R^n}\frac{v^q(y)dy}{|x-y|^{n-\alpha}}, \\ v(x)=\displaystyle\int_{R^n}\frac{u^p(y)dy}{|x-y|^{n-\alpha}} \end{cases} \tag{5.6}$$

and the energy integrals $\|u\|_{L^{p+1}(R^n)}$ *and* $\|v\|_{L^{q+1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-\alpha}{n}. \tag{5.7}$$

(4) The Wolff type system

$$\begin{cases} u(x)=W_{\beta,\gamma}(v^q)(x), \\ v(x)=W_{\beta,\gamma}(u^p)(x). \end{cases} \tag{5.8}$$

and the energy integrals $\|u\|_{L^{p+\gamma-1}(R^n)}$ *and* $\|v\|_{L^{q+\gamma-1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$\frac{1}{p+\gamma-1}+\frac{1}{q+\gamma-1}=\frac{n-\beta\gamma}{n(\gamma-1)}. \tag{5.9}$$

In addition, (5.8) and the energy integrals $\|u\|_{L^{p+1}(R^n)}$ *and* $\|v\|_{L^{q+1}(R^n)}$ *are invariant under the scaling transforms, if and only if*

$$p=q \quad or \quad \gamma=2. \tag{5.10}$$

Proof. (1) Take the scaling transforms

$$u_\mu(x) = \mu^{\sigma_1} u(\mu x), \quad v_\mu(x) = \mu^{\sigma_2} v(\mu x). \tag{5.11}$$

Then

$$-\Delta_x u_\mu(x) = \mu^{\sigma_1+2}[-\Delta_y u(y)] = \mu^{\sigma_1+2} v^q(y) = \mu^{\sigma_1+2-q\sigma_2} v_\mu^q(x),$$

and similarly,

$$-\Delta v_\mu = \mu^{\sigma_2+2-p\sigma_1} u_\mu^p.$$

On the other hand,

$$\int_{R^n} u_\mu^{p+1}(x)dx = \mu^{\sigma_1(p+1)} \int_{R^n} u^{p+1}(\mu x)dx = \mu^{\sigma_1(p+1)-n} \int_{R^n} u^{p+1}(y)dy,$$

and similarly,

$$\int_{R^n} v_\mu^{q+1}(x)dx = \mu^{\sigma_2(q+1)-n} \int_{R^n} v^{q+1}(y)dy.$$

Clearly, (5.1) is invariant if and only if

$$\sigma_1 + 2 = q\sigma_2, \quad \sigma_2 + 2 = p\sigma_1.$$

Energy integrals are invariant if and only if

$$\sigma_1(p+1) = n, \quad \sigma_2(q+1) = n.$$

Eliminate σ_1 and σ_2. Then

$$\frac{pq-1}{(p+1)(q+1)} = \frac{2}{n}.$$

This is equivalent to (5.2).

(2) In view of (5.11), we have

$$-\Delta_\gamma u_\mu(x) = \mu^{\sigma_1(\gamma-1)+\gamma}[-\Delta_\gamma u(\mu x)] = \mu^{\sigma_1(\gamma-1)+\gamma-q\sigma_2} v_\mu^q(x).$$

Similarly,

$$-\Delta_\gamma v_\lambda = \mu^{\sigma_2(\gamma-1)+\gamma-p\sigma_1} u_\mu^p(x).$$

In addition,

$$\begin{aligned} \int_{R^n} u_\mu^{p+\gamma-1}(x)dx &= \mu^{\sigma_1(p+\gamma-1)} \int_{R^n} u^{p+\gamma-1}(\mu x)dx \\ &= \mu^{\sigma_1(p+\gamma-1)-n} \int_{R^n} u^{p+\gamma-1}(y)dy, \end{aligned}$$

and similarly,

$$\int_{R^n} v_\mu^{q+\gamma-1}(x)dx = \mu^{\sigma_2(q+\gamma-1)-n} \int_{R^n} v^{q+\gamma-1}(y)dy.$$

Eq. (5.3) is invariant if and only if

$$\sigma_1(\gamma-1)+\gamma-q\sigma_2=0, \quad \sigma_2(\gamma-1)+\gamma-p\sigma_1=0.$$

Namely,

$$\sigma_1=\frac{\gamma(q+\gamma-1)}{pq-(\gamma-1)^2}, \quad \sigma_2=\frac{\gamma(p+\gamma-1)}{pq-(\gamma-1)^2}. \tag{5.12}$$

Energy integrals $\|u\|_{L^{p+\gamma-1}(R^n)}$ and $\|v\|_{L^{q+\gamma-1}(R^n)}$ are invariant if and only if

$$\sigma_1(p+\gamma-1)-n=0, \quad \sigma_2(q+\gamma-1)-n=0.$$

Eliminating σ_1 and σ_2, we obtain (5.4).

Similarly, $\|u\|_{L^{p+1}(R^n)}$ and $\|v\|_{L^{q+1}(R^n)}$ are invariant if and only if

$$\sigma_1=\frac{n}{p+1}, \quad \sigma_2=\frac{n}{q+1}.$$

Combining with (5.12), we see $(q+1)(p+\gamma-1)=(p+1)(q+\gamma-1)$. This is equivalent to $(p-q)(\gamma-2)=0$. Thus, (5.5) is the sufficient and necessary condition.

(3) Noting (5.11), we have

$$\begin{aligned} v_\mu(x) &= \mu^{\sigma_2}\int_{R^n}\frac{u^p(y)dy}{|\mu x-y|^{n-\alpha}}=\mu^{\sigma_2}\int_{R^n}\frac{\mu^n u^p(\mu z)dz}{|\mu(x-z)|^{n-\alpha}} \\ &= \mu^{\sigma_2}\int_{R^n}\frac{\mu^{n-p\sigma_1}u_\mu^p(z)dz}{\mu^{n-\alpha}|x-z|^{n-\alpha}}=\mu^{\sigma_2-p\sigma_1+\alpha}\int_{R^n}\frac{u_\mu^p(y)dy}{|x-y|^{n-\alpha}}. \end{aligned}$$

Similarly,

$$u_\mu(x)=\mu^{\sigma_1-q\sigma_2+\alpha}\int_{R^n}\frac{v_\mu^q(y)dy}{|x-y|^{n-\alpha}}.$$

Thus, u_μ, v_μ still solve (5.6) if and only if

$$\sigma_1+\alpha=q\sigma_2, \quad \sigma_2+\alpha=p\sigma_1.$$

By the same calculation in (1), energy integrals are invariant if and only if

$$\sigma_1(p+1)=n, \quad \sigma_2(q+1)=n.$$

Eliminating σ_1 and σ_2, we deduce (5.7).

(4) Noting (5.11), we have

$$\begin{aligned} v_\mu(x) &= \mu^{\sigma_2}\int_0^\infty\left(\frac{\int_{B_t(\mu x)}u^p(y)dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ &= \mu^{\sigma_2}\int_0^\infty\left(\frac{\int_{B_t(\mu x)}u^p(\mu z)d(\mu z)}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ &= \mu^{\sigma_2}\int_0^\infty\left(\frac{\int_{B_s(x)}\mu^{n-p\sigma_1}u_\mu^p(z)dz}{(\mu s)^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{ds}{s} \\ &= \mu^{\sigma_2+\frac{\beta\gamma-p\sigma_1}{\gamma-1}}\int_0^\infty\left(\frac{\int_{B_s(x)}u_\mu^p(z)dz}{s^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{ds}{s}. \end{aligned}$$

Similarly,

$$u_\mu(x) = \mu^{\sigma_1 + \frac{\beta\gamma - q\sigma_2}{\gamma-1}} \int_0^\infty \left(\frac{\int_{B_s(x)} v_\mu^q(z)dz}{s^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{ds}{s}.$$

Thus, u_μ, v_μ still solve (5.8) if and only if

$$(\gamma-1)\sigma_1 + \beta\gamma = q\sigma_2, \quad (\gamma-1)\sigma_2 + \beta\gamma = p\sigma_1.$$

By the same calculation in (2), energy integrals $\|u\|_{L^{p+\gamma-1}(R^n)}$ and $\|v\|_{L^{q+\gamma-1}(R^n)}$ are invariant if and only if

$$\sigma_1(p+\gamma-1) = n, \quad \sigma_2(q+\gamma-1) = n.$$

Eliminating σ_1 and σ_2, we deduce (5.9).

By the same argument in (2), (5.10) is another corresponding sufficient and necessary condition. $\square$

5.2 Existence and the critical conditions

In this subsection, we first show that (5.2) is the critical condition of the existence of the finite energy solution of (5.1). We call the positive classical solutions u, v of (5.1) *finite energy solutions*, if $u \in L^{p+1}(R^n) \cap L^{2^*}(R^n)$, and $v \in L^{q+1}(R^n) \cap L^{2^*}(R^n)$.

Theorem 5.2. *The system (5.1) has a pair of finite energy solutions* (u, v) *if and only if (5.2) holds.*

Proof. Serrin and Zou [46] proved the existence if (5.2) is true. Next, we will deduce (5.2) from the existence. Denote $B_R(0)$ by B. According to Proposition 5.1 in [45] (or cf. Lemma 2.6 in [48]), the solutions u, v satisfy the Pohozaev type identity

$$\begin{aligned}
&\left(\frac{n}{p+1} - a_1\right)\int_B u^{p+1}dx + \left(\frac{n}{q+1} - a_2\right)\int_B v^{q+1}dx \\
&= R^n \int_{S^{n-1}} \left(\frac{u^{p+1}}{p+1} + \frac{v^{q+1}}{q+1}\right)ds + R^{n-1}\int_{S^{n-1}} (a_1 u\partial_r v + a_2 v\partial_r u)ds \\
&\quad + R^n \int_{S^{n-1}} \left(\partial_r u \partial_r v - \frac{\partial_\theta u \partial_\theta v}{R^2}\right)ds,
\end{aligned} \tag{5.13}$$

where $a_2, a_2 \in R$ satisfy $a_1 + a_2 = n-2$. Since u, v are finite energy solutions, we know $\nabla u, \nabla v \in L^2(R^n)$ by an analogous argument of Theorem 4.4. Using Proposition 2.1 and the Young inequality, we can find $R_j \to \infty$, such that all the terms in the right hand side converge to zero. Letting $R = R_j \to \infty$ in the Pohozaev identity above, we obtain

$$\left(\frac{n}{p+1} - a_1\right)\int_{R^n} u^{p+1}dx + \left(\frac{n}{q+1} - a_2\right)\int_{R^n} v^{q+1}dx = 0$$

for any a_1, a_2 as long as $a_1 + a_2 = n - 2$. Take $a_2 = \frac{n}{q+1}$, then

$$(\frac{n}{p+1} - a_1)\int_{R^n} u^{p+1}dx = 0.$$

This implies $0 = \frac{n}{p+1} - a_1 = \frac{n}{p+1} - (n-2-a_2) = \frac{n}{p+1} - (n-2) + \frac{n}{q+1}$. So (5.2) is verified. □

Next, we consider the HLS type system. Since (5.6) is the Euler-Lagrange system of the extremal functions of the HLS inequality which implies $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$, we naturally call such solutions (belonging to $L^{p+1}(R^n) \times L^{q+1}(R^n)$) of (5.6) as *finite energy solutions*.

Theorem 5.3. *The HLS type system (5.6) has the finite energy solutions if and only if (5.7) holds.*

Proof. Sufficiency. Clearly, the extremal functions of the HLS inequality are the finite energy solutions. Lieb [35] obtained the existence of those extremal functions.

Necessity. The Pohozaev type identity in integral forms is used here.

For any $\mu \neq 0$, there holds

$$v(\mu x) = \int_{R^n} \frac{u^p(y)dy}{|\mu x - y|^{n-\alpha}} = \mu^\alpha \int_{R^n} \frac{u^p(\mu z)dz}{|x-z|^{n-\alpha}}.$$

Differentiate both sides with respect to μ and let $\mu = 1$. Then,

$$x \cdot \nabla v = \alpha v + \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x-z|^{n-\alpha}}. \tag{5.14}$$

According to Remark 1.2 (1) (or cf. Theorem 1 in [6]), if $p, q \leq \frac{\alpha}{n-\alpha}$, (5.6) has no any positive solution. Therefore, (u,v) solves (5.6) implies $p, q > \frac{\alpha}{n-\alpha}$. Similar to the derivation of (4.18), if follows

$$R\int_{\partial B_R} \frac{u^p(z)ds}{|x-z|^{n-\alpha}} \to 0, \quad R\int_{\partial B_R} \frac{v^q(z)ds}{|x-z|^{n-\alpha}} \to 0,$$

when $R = R_j \to \infty$. Thus, integrating by parts, we obtain

$$\int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x-z|^{n-\alpha}} = -nv - (n-\alpha)\int_{R^n} \frac{(z \cdot (x-z))u^p(z)}{|x-z|^{n-\alpha+2}}dz.$$

Multiplying (5.14) by $v^q(x)$ we get

$$\begin{aligned}
&\int_{R^n} v^q(x)(x \cdot \nabla v(x))dx \\
&= \alpha \int_{R^n} v^{q+1}(x)dx + \int_{R^n} v^q(x)dx \int_{R^n} \frac{z \cdot \nabla u^p(z)dz}{|x-z|^{n-\alpha}} \\
&= \alpha \int_{R^n} v^{q+1}(x)dx - n\int_{R^n} v^{q+1}(x)dx \\
&\quad -(n-\alpha)\int_{R^n}\int_{R^n} \frac{(z \cdot (x-z))v^q(x)u^p(z)}{|x-z|^{n-\alpha+2}}dzdx.
\end{aligned}$$

Similarly, there also holds

$$\begin{aligned}
&\int_{R^n} u^p(x)(x\cdot\nabla u(x))dx\\
&=(\alpha-n)\int_{R^n} u^{p+1}(x)dx-(n-\alpha)\int_{R^n}\int_{R^n}\frac{(z\cdot(x-z))v^q(z)u^p(x)}{|x-z|^{n-\alpha+2}}dzdx\\
&=(\alpha-n)\int_{R^n} u^{p+1}(x)dx-(n-\alpha)\int_{R^n}\int_{R^n}\frac{(x\cdot(z-x))v^q(x)u^p(z)}{|x-z|^{n-\alpha+2}}dzdx.
\end{aligned}$$

By virtue of $z\cdot(x-z)+x\cdot(z-x)=-|x-z|^2$, it follows that

$$\begin{aligned}
&\int_{R^n} v^q(x)(x\cdot\nabla v(x))dx+\int_{R^n} u^p(x)(x\cdot\nabla u(x))dx\\
&=(\alpha-n)(\int_{R^n} v^{q+1}(x)dx+\int_{R^n} u^{p+1}(x)dx)\\
&\quad+(n-\alpha)\int_{R^n}\int_{R^n}\frac{v^q(x)u^p(z)}{|x-z|^{n-\alpha}}dzdx.
\end{aligned}$$

On the other hand, integrating by parts leads to

$$\begin{aligned}
\int_{R^n} v^q(x)(x\cdot\nabla v(x))dx &=\frac{1}{q+1}\int_{R^n}(x\cdot\nabla v^{q+1}(x))dx\\
&=\frac{-n}{q+1}\int_{R^n} v^{q+1}(x)dx
\end{aligned}$$

and similarly $\int_{R^n} u^p(x)(x\cdot\nabla u(x))dx=\frac{-n}{p+1}\int_{R^n} u^{p+1}(x)dx$. Inserting these into the result above, we deduce that

$$\begin{aligned}
&-\frac{n}{q+1}\int_{R^n} v^{q+1}(x)dx-\frac{n}{p+1}\int_{R^n} u^{p+1}(x)dx\\
&=(\alpha-n)(\int_{R^n} v^{q+1}(x)dx+\int_{R^n} u^{p+1}(x)dx)\\
&\quad+(n-\alpha)\int_{R^n}\int_{R^n}\frac{v^q(x)u^p(z)}{|x-z|^{n-\alpha}}dzdx.
\end{aligned}$$

From (5.6), it follows that

$$\begin{aligned}
&\int_{R^n} v^{q+1}(x)dx=\int_{R^n} v^q(x)dx\int_{R^n}\frac{u^p(y)dy}{|x-y|^{n-\alpha}}\\
&=\int_{R^n} u^p(x)dx\int_{R^n}\frac{v^q(y)dy}{|x-y|^{n-\alpha}}=\int_{R^n} u^{p+1}(x)dx.
\end{aligned}$$

Substituting this into the result above yields

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-\alpha}{n}.$$

Theorem 5.3 is proved. $\square$

Corollary 5.4. *Let $k \in [1, n/2)$ be an integer and $pq > 1$. The $2k$-order system*

$$\begin{cases} (-\Delta)^k u = v^q, \\ (-\Delta)^k v = u^p, \end{cases}$$

has a pair of finite energy positive solutions (u, v), then

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2k}{n}.$$

Proof. Since $pq > 1$, [37] proved that the solutions u, v of this system satisfy $(-\Delta)^i u \geq 0$, $(-\Delta)^i v \geq 0$ for $i = 1, 2, \cdots, k-1$. Thus, this system is equivalent to the integral system (5.6) with $\alpha = 2k$ (cf. [8]). According to Theorem 5.3, we can also derive the conclusion. □

6 Infinite energy solutions

6.1 Existence in supercritical case

For semilinear Lane-Emden equation (1.5), Li [32] obtained a positive solution with the slow decay rate

$$u(x) = O(|x|^{-\frac{2}{p-1}}), \quad when\ |x| \to \infty.$$

According to Corollary 1.3, it is not the finite energy solution.

In this section, we prove that there also exists an infinite energy solution for bi-Laplace equation in the supercritical case $p > \frac{p+4}{p-4}$.

Clearly,

$$(-\Delta)^2 u = u^p, \quad in\ R^n,$$

is equivalent to

$$\begin{cases} -\Delta u = v, \\ -\Delta v = u^p. \end{cases}$$

We search the positive solutions with radial structures. The existence can be implied by the following argument.

Theorem 6.1. *The following ODE system*

$$\begin{cases} -(u'' + \frac{n-1}{r}u') = v, \quad -(v'' + \frac{n-1}{r}v') = u^p, \quad r > 0 \\ u'(0) = v'(0) = 0, \quad u(0) = 1, \quad v(0) = a, \end{cases} \tag{6.1}$$

has entire solutions satisfying $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0$.

Proof. Here we use the shooting method.

We denote the solutions of (6.1) by $u_a(r), v_a(r)$.

Step 1. By the standard contraction argument, we can see the local existence.

Step 2. We claim that for $a \geq 4n$, there exists $R \in (0, 1]$ such that $u_a(r), v_a(r) > 0$ for $r \in [0, R)$ and $u_a(R) = 0$.

In fact, from (6.1) we obtain $u_a' < 0$ which implies $u_a(r) \le u_a(0) = 1$, and

$$v_a(r) = v_a(0) - \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} u_a^p(s) ds d\tau \ge a - \frac{r^2}{2n} \ge \frac{a}{2}$$

for $r \in [0,1]$. Therefore,

$$u_a(r) = u_a(0) - \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} v_a(s) ds d\tau \le 1 - \frac{ar^2}{4n}.$$

This proves that for $a \ge 4n$, we can find $R \in (0,1]$ such that $u_a(r), v_a(r) > 0$ for $r \in (0,R)$ and $u_a(R) = 0$.

Step 3. We claim that for $0 < a < \varepsilon_0 = \frac{1}{n2^{1+p}}$, there exists $R \in (0,1]$, such that $u_a(r), v_a(r) > 0$ for $r \in [0,R)$ and $v_a(R) = 0$.

In fact,

$$u_a(r) \ge 1 - \frac{\varepsilon_0 r^2}{2n} \ge \frac{1}{2},$$

for $r \in (0,1)$. Therefore,

$$v_a(r) < \varepsilon_0 - \frac{1}{2^p}\frac{r^2}{2n}.$$

This proves that for $a < \varepsilon_0$, we can find $R \in (0,1]$ such that $u_a(r), v_a(r) > 0$ for $r \in (0,R)$ and $v_a(R) = 0$.

Step 4. Let $\underline{a} = \sup \underline{S}$, where

$$\underline{S} := \{\varepsilon; \exists R_a > 0, \text{ such that } u_a(r) > 0, v_a(r) \ge 0, \text{ for } r \in [0,R_a], v_a(R_a) = 0\}.$$

Clearly, $\underline{S} \ne \emptyset$ by virtue of $\varepsilon_0 \in \underline{S}$. Noting $\varepsilon \le a_0$ for $\varepsilon \in \underline{S}$, we see the existence of $\underline{a}$.

Step 5. Write $\bar{u}(r) = u_{\underline{a}}(r)$ and $\bar{v}(r) = v_{\underline{a}}(r)$. We claim that $\bar{u}(r), \bar{v}(r) > 0$ for $r \in [0,\infty)$, and hence they are entire positive solutions of (6.1).

Otherwise, there exists $\bar{R} > 0$ such that $\bar{u}(r), \bar{v}(r) > 0$ for $r \in (0,\bar{R})$ and one of the following consequences holds:

(i) $\bar{u}(\bar{R}) = 0$, $\bar{v}(\bar{R}) > 0$;
(ii) $\bar{v}(\bar{R}) = 0$, $\bar{u}(\bar{R}) > 0$;
(iii) $\bar{u}(\bar{R}) = 0$, $\bar{v}(\bar{R}) = 0$.

We deduce the contradictions from three consequences above.

(i) By C^1-continuous dependence of u_a, v_a in a, and the fact $\bar{u}'(\bar{R}) < 0$, we see that for all $|a - \underline{a}|$ small, there exists $R_a > 0$ such that

$$\begin{array}{l} \bar{u}(r), \bar{v}(r) > 0, \quad for\ r \in (0,R_a); \\ \bar{u}(R_a) = 0, \quad \bar{v}(R_a) > 0. \end{array}$$

This contradicts with the definition of $\underline{a}$.

(ii) Similarly, for $|a - \underline{a}|$ small, there exists $R_a > 0$ such that

$$\begin{array}{l} \bar{u}(r), \bar{v}(r) > 0, \quad for\ r \in (0,R_a); \\ \bar{u}(R_a) > 0, \quad \bar{v}(R_a) = 0. \end{array}$$

This implies that $\underline{a}+\delta \in \underline{S}$ for some $\delta > 0$, which contradicts with the definition of $\underline{a}$.

(iii) The consequence implies that $u(x) = \bar{u}(|x|)$ and $v(x) = \bar{v}(|x|)$ are solutions of the system

$$\begin{cases} -\Delta u = v, \quad -\Delta v = u^p, \; in \; B_R, \\ u, v > 0 \; in \; B_R, \quad u = v = 0 \; on \; \partial B_R. \end{cases} \tag{6.2}$$

It is impossible by the Pohozaev identity proved later (cf. Theorem 6.3).

All the contradictions show that our claim is true. Thus, the entire positive solutions exist.

Step 6. We claim $\lim_{r\to\infty} \bar{u}(r), \bar{v}(r) = 0$.

Eq. (6.1) implies $\bar{u}' < 0$ and $\bar{v}' < 0$ for $r > 0$. So $\bar{u}$ and $\bar{v}$ are decreasing positive solutions, and $\lim_{r\to\infty} \bar{u}(r)$, $\lim_{r\to\infty} \bar{v}(r)$ exist.

If there exists $c > 0$ such that $\bar{v}(r) \geq c$ for $r > 0$, then (6.1) shows that $\bar{u}$ satisfies

$$u'' + \frac{n-1}{r}u' \leq -c.$$

Integrating twice yields

$$\bar{u}(r) \leq \bar{u}(0) - \frac{cr^2}{2n}$$

for $r > 0$. It is impossible since $\bar{u}$ is a entire positive solution. This shows that $\bar{v} \to 0$ when $r \to \infty$.

Similarly, u has the same property. □

Remark 6.1.

1. When $k \in (2, n/2)$ is an integer, the existence of the $2k$-order PDEs in the supercritical cases is rather challenged. Recently, Li [29] applied the shooting method and the analysis of the target map via the degree theory to obtain the existence results for both (1.6) and (1.2) in Theorem 1.1 in the supercritical cases.

2. In the critical case $p = \frac{n+\alpha}{n-\alpha}$, (1.10) with $\alpha = 2k$ is a solution of (1.6). For the system (1.2), the critical condition $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{\alpha}{n}$ leads to $pq > 1$. The argument in Corollary 5.4 shows the equivalence between (1.2) and the HLS type system (5.6). Therefore, the existence of (1.2) is implied by the sufficiency of Theorem 5.3.

3. In the subcritical case $p < \frac{n+\alpha}{n-\alpha}$, the nonexistence of positive solutions of (1.7) had been proved (cf. [1], [9] and [51]). On the other hand, by the equivalence between (1.6) and (1.7) (cf. [8] and [11]), we also see that (1.6) does not exist any positive solution. As regards the nonexistence for the system (1.2) (or (1.1)), it is the Lane-Emden conjecture (or the HLS conjecture) (cf. [2] and [48]).

6.2 Nonexistence in bounded domain

In this subsection, we give the Pohozaev identity which the proof of Theorem 6.1 needs. In fact, we can give more general ones which imply nonexistence of positive solutions of the following $2k$-order PDE (1.6)

$$(-\Delta)^k u = u^p, \quad k \geq 1, \ u > 0$$

with the supercritical exponent $p > \frac{n+2k}{n-2k}$ in any bounded domain. The argument can help to prove the existence results in R^n.

Note. Seeing here, we recall another related fact: in the subcritical case, (1.6) has positive solutions in a bounded domain. In general, the variational methods works now. However, it has no positive solution in R^n (cf. Remark 6.1(3)).

Proposition 6.2. *Let $D \subset R^n$ be a bounded domain. Assume that u_j ($j = 1, 2, \cdots, k$) solve the following boundary value problem*

$$\begin{cases} -\Delta u_1 = u_2, \quad -\Delta u_2 = u_3, \quad \cdots, \\ -\Delta u_{k-1} = u_k, \quad -\Delta u_k = u_{k+1} := u_1^p, \quad in\ D, \\ u_1 = u_2 = \cdots = u_k = 0, \quad on\ \partial D. \end{cases} \tag{6.3}$$

Then

$$\begin{aligned} &\int_D u_1^{p+1} dx = \int_D u_j u_{k+2-j} dx, \quad for\ j = 1, 2, \cdots, k; \\ &\int_D u_1^{p+1} dx = \int_D \nabla u_j \nabla u_{k+1-j} dx, \quad for\ j = 1, 2, \cdots, k. \end{aligned} \tag{6.4}$$

Proof. Applying the boundary value condition, from (6.3) we obtain

$$\begin{aligned} &\int_D u_1^{p+1} dx = -\int_D u_1 \Delta u_k dx = \int_D \nabla u_1 \nabla u_k dx \\ &= -\int_D u_k \Delta u_1 dx = \int_D u_2 u_k dx = -\int_D u_2 \Delta u_{k-1} dx \\ &= \int_D \nabla u_2 \nabla u_{k-1} dx = \int_D u_3 u_{k-1} dx = \cdots \\ &= \int_D \nabla u_j \nabla u_{k+1-j} dx = \int_D u_j u_{k+2-j} dx. \end{aligned}$$

This result implies (6.4). □

Theorem 6.3. *Let $D \subset R^n$ be a bounded star-shaped domain. If*

$$p \geq \frac{n+2k}{n-2k}, \tag{6.5}$$

then the following Navier boundary value problem has no positive radial solution in $C^{2k}(D) \cap C^{2k-1}(\bar{D})$

$$\begin{cases} (-\Delta)^k u = u^p \quad in \quad D, \\ u = \Delta u = \cdots = \Delta^{k-1} u = 0 \quad on \quad \partial D. \end{cases} \tag{6.6}$$

Proof. Clearly, $u = u_1$ satisfies

$$\begin{cases} -\Delta u_1 = u_2, \ -\Delta u_2 = u_3, \ \cdots, \\ -\Delta u_{k-1} = u_k, \ -\Delta u_k = u_{k+1} := u_1^p, \quad in\ D, \\ u_1 = u_2 = \cdots = u_k = 0, \quad on\ \partial D. \end{cases}$$

By the maximum principle, from $-\Delta u_k = u_1^p > 0$ and $u_k|_{\partial D} = 0$, we see $u_k > 0$ in D. By the same way, we also deduce by induction that

$$u_j > 0 \quad in\ D, \quad j = 1, 2, \cdots, k. \tag{6.7}$$

Multiplying the j-th equation by $(x \cdot \nabla u_{k+1-j})$, we have

$$\begin{aligned} &-\int_{\partial D} (x \cdot \nu)\partial_\nu u_j \partial_\nu u_{k+1-j} ds + \int_D \nabla u_j \nabla u_{k+1-j} dx \\ &+\int_D x \cdot \nabla u_{jx_i} (u_{k+1-j})_{x_i} dx = \int_D u_{j+1}(x \cdot \nabla u_{k+1-j}) dx \end{aligned} \tag{6.8}$$

for $j = 1, 2, \cdots, k$, where ν is the unit outward normal vector on ∂D.

Integrating by parts, we can see that

$$\begin{aligned} &\int_D x \cdot (u_{jx_i} \nabla (u_{k+1-j})_{x_i} + (u_{k+1-j})_{x_i} \nabla u_{jx_i}) dx \\ &= \int_D x \cdot \nabla(\nabla u_j \nabla u_{k+1-j}) dx \\ &= \int_{\partial D} (x \cdot \nu)\partial_\nu u_j \partial_\nu u_{k+1-j} ds - n \int_D \nabla u_j \nabla u_{k+1-j} dx, \end{aligned}$$

Combining the results of (6.8) with j and $k+1-j$, and using the result above, we deduce that, for $j = 1, 2, 3, \cdots, k$,

$$\begin{aligned} &-\int_{\partial D} (x \cdot \nu)\partial_\nu u_j \partial_\nu u_{k+1-j} ds + (2-n) \int_D \nabla u_j \nabla u_{k+1-j} dx \\ &= \int_D u_{k+2-j}(x \cdot \nabla u_j) dx + \int_D u_{j+1}(x \cdot \nabla u_{k+1-j}) dx. \end{aligned} \tag{6.9}$$

Integrating by parts, we also see that for $j = 2, 3, \cdots, k$,

$$\begin{aligned} &\int_D x \cdot (u_j \nabla u_{k+2-j} + u_{k+2-j} \nabla u_j) dx \\ &= \int_D x \cdot \nabla(u_j u_{k+2-j}) dx = -n \int_D u_j u_{k+2-j} dx. \end{aligned}$$

Summing j from 1 to k in (6.9) and using the result above, we obtain

$$\begin{aligned} &\frac{2-n}{2} \int_D (\nabla u_1 \nabla u_k + \nabla u_2 \nabla u_{k-1} + \cdots + \nabla u_k \nabla u_1) dx \\ &+\frac{n}{2} \int_D (u_2 u_k + u_3 u_{k-1} + \cdots + u_k u_2) dx - \int_D u_{k+1}(x \cdot \nabla u_1) dx \\ &= \int_{\partial D} (x \cdot \nu)[(\partial_\nu u_1 \partial_\nu u_k + \partial_\nu u_2 \partial_\nu u_{k-1} + \cdots + \partial_\nu u_k \partial_\nu u_1). \end{aligned}$$

By virtue of (6.7) and the boundary value condition, the Hopf lemma shows that $\partial_\nu u_j < 0$ on ∂D for $j = 1, 2, \cdots, k$. Noting D is star-shaped, we know that all terms in the right hand side of the result above are positive. Namely,

$$\begin{aligned}
&\frac{2-n}{2}\int_D(\nabla u_1\nabla u_k + \nabla u_2\nabla u_{k-1} + \cdots + \nabla u_k\nabla u_1)dx \\
&+\frac{n}{2}\int_D(u_2u_k + u_3u_{k-1} + \cdots + u_ku_2)dx + \frac{n}{p+1}\int_D u^{p+1}dx > 0.
\end{aligned} \tag{6.10}$$

Inserting (6.4) into (6.10), we have

$$\frac{n}{p+1} + \frac{k(2-n)}{2} + \frac{n(k-1)}{2} > 0.$$

This contradicts (6.5). □

The following result is necessary to prove Theorem 1.5 (2) (cf. [29]).

Theorem 6.4. *Let $D \subset R^n$ be a bounded star-shaped domain. If*

$$\frac{1}{p+1} + \frac{1}{q+1} \le \frac{n-2k}{n}, \tag{6.11}$$

then the following Navier boundary value problem has no positive radial solution in $C^{2k}(D) \cap C^{2k-1}(\bar{D})$

$$\begin{cases} (-\Delta)^k u = v^q, \quad (-\Delta)^k v = u^p \quad \textit{in } D, \\ u = \Delta u = \cdots = \Delta^{k-1}u = 0 \quad \textit{on} \quad \partial D, \\ v = \Delta v = \cdots = \Delta^{k-1}v = 0 \quad \textit{on} \quad \partial D. \end{cases} \tag{6.12}$$

Proof. Clearly, the solutions $u_1(= u)$ and $v_1(= v)$ of (6.12) satisfy

$$\begin{cases} -\Delta u_1 = u_2, -\Delta u_2 = u_3, \cdots, -\Delta u_k = u_{k+1} := v^q, \quad \textit{in } D, \\ -\Delta v_1 = v_2, -\Delta v_2 = v_3, \cdots, -\Delta v_k = v_{k+1} := u^p, \quad \textit{in } D, \\ u_1 = u_2 = \cdots = u_k = 0 \quad \textit{on} \quad \partial D, \\ v_1 = v_2 = \cdots = v_k = 0 \quad \textit{on} \quad \partial D. \end{cases}$$

Multiply $-\Delta u_j = u_{j+1}$ and $-\Delta v_j = v_{j+1}$ by $(x \cdot \nabla v_{k+1-j})$ and $(x \cdot \nabla u_{k+1-j})$, respectively. Integrating by parts yields

$$\begin{aligned}
&-\int_{\partial D}(x\cdot\nu)\partial_\nu u_j\partial_\nu v_{k+1-j}ds + \int_D \nabla u_j\nabla v_{k+1-j}dx \\
&+\int_D[x\cdot\nabla(v_{k+1-j})_{x_i}]u_{jx_i}dx = \int_D u_{j+1}(x\cdot\nabla v_{k+1-j})dx
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
&-\int_{\partial D}(x\cdot\nu)\partial_\nu v_j\partial_\nu u_{k+1-j}ds + \int_D \nabla v_j\nabla u_{k+1-j}dx \\
&+\int_D[x\cdot\nabla(u_{k+1-j})_{x_i}]v_{jx_i}dx = \int_D v_{j+1}(x\cdot\nabla u_{k+1-j})dx.
\end{aligned} \tag{6.14}$$

Adding the $(k+1-j)$-th (6.13) and the j-th (6.14) together leads to

$$\begin{aligned} &-\int_{\partial D}(x\cdot\nu)\partial_\nu v_1\partial_\nu u_k ds+(2-n)\int_D \nabla v_1\nabla u_k dx \\ &=-\frac{n}{q+1}\int_D v_1^{q+1}dx+\int_D v_2(x\cdot\nabla u_k)dx, \end{aligned} \tag{6.15}$$

$$\begin{aligned} &-\int_{\partial D}(x\cdot\nu)\partial_\nu u_1\partial_\nu v_k ds+(2-n)\int_D \nabla u_1\nabla v_k dx \\ &=-\frac{n}{p+1}\int_D u_1^{q+1}dx+\int_D u_2(x\cdot\nabla v_k)dx, \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} &-\int_{\partial D}(x\cdot\nu)\partial_\nu v_j\partial_\nu u_{k+1-j} ds+(2-n)\int_D \nabla v_j\nabla u_{k+1-j} dx \\ &=\int_D [v_{j+1}(x\cdot\nabla u_{k+1-j})+u_{k+2-j}(x\cdot\nabla v_j)]dx. \end{aligned} \tag{6.17}$$

Summing j from 1 to k, by (6.15), (6.16) and (6.17) we deduce that

$$\begin{aligned} &n\int_D(\frac{u_1^{p+1}}{p+1}+\frac{v_1^{q+1}}{q+1})dx+n\int_D\sum_{j=2}^k v_j u_{k+2-j}dx \\ &+(2-n)\int_D\sum_{j=1}^k \nabla v_j\nabla u_{k+1-j}dx \\ &=\int_{\partial D}(x\cdot\nu)\sum_{j=1}^k\partial_\nu v_j\partial_\nu u_{k+1-j}ds>0. \end{aligned}$$

Similar to Proposition 6.2, it also follows

$$\int_D u_1^{p+1}dx=\int_D v_1^{q+1}dx=\int_D\nabla v_j\nabla u_{k+1-j}dx=\int_D v_l u_{k+2-l}$$

for $1\le j\le k$ and $2\le l\le k$.

Combining two results above, we have

$$\frac{n}{p+1}+\frac{n}{q+1}+n(k-1)+(2-n)k>0,$$

which contradicts (6.11). $\square$

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References

[1] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

[2] G. Caristi, L. D'Ambrosio, E. Mitidieri, *Representation formulae for solutions to some classes of higher order systems and related Liouville theorems*, Milan J. Math., **76**, (2008), 27–67.

[3] A. Chang, P. Yang, *On uniqueness of an n-th order differential equation in conformal geometry*, Math. Res. Lett., **4** (1997), 91–102.

[4] W. Chen, C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J., **63** (1991), 615–622.

[5] W. Chen, C. Li, *A priori estimates for prescribing scalar curvature equations*, Ann. of Math., **145** (1997), 547–564.

[6] W. Chen, C. Li, *An integral system and the Lane-Emden conjecture*, Discrete Contin. Dyn. Syst., **24** (2009), 1167–1184.

[7] W. Chen, C. Li, *Radial symmetry of solutions for some integral systems of Wolff type*, Discrete Contin. Dyn. Syst., **30** (2011), 1083–1093.

[8] W. Chen, C. Li, *Super polyharmonic property of solutions for PDE systems and its applications*, Commun. Pure Appl. Anal., (2013), in press. arXiv:1110.2539v1.

[9] W. Chen, C. Li, B. Ou, *Qualitative properties of solutions for an integral equation*, Discrete Contin. Dyn. Syst., **12** (2005), 347–354.

[10] W. Chen, C. Li, B. Ou, *Classification of solutions for a system of integral equations*, Comm. Partial Differential Equations, **30** (2005), 59–65.

[11] W. Chen, C. Li, B. Ou, *Classification of solutions for an integral equation*, Commun. Pure Appl. Math., **59** (2006), 330–343.

[12] M. Franca, *Classification of positive solutions of p-Laplace equation with a growth term*, Archivum Mathematicum, **40** (2004), 415–434.

[13] F. Gazzola, *Critical exponents which relate embedding inequalities with quasilinear elliptic operator*, Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations, May 24-27, 2002, Wilmington, NC, USA, pp.327–335.

[14] B. Gidas, W.-M. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in R^n* (collected in the book *Mathematical Analysis and Applications*, which is vol. 7a of the book series *Advances in Mathematics. Supplementary Studies*, Academic Press, New York, 1981.)

[15] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., **34** (1981), 525–598.

[16] M. Guedda, L. Veron, *Local and global properties of solutions of quasilinear elliptic equations*, J. Differential Equations, **76** (1988), 159–189.

[17] C. Gui, *On positive entire solutions of the elliptic equation* $\Delta u+K(x)u^p=0$ *and its applications to Riemannian geometry*, Proc. Roy. Soc. Edinburgh Sect. A, **126** (1996), 225–237.

[18] F. Hang, *On the integral systems related to Hardy-Littlewood-sobolev inequality*, Math. Res. Lett., **14** (2007), 373–383.

[19] L.I. Hedberg and T. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenobel), **33**, (1983), 161–187.

[20] C. Jin, C. Li, *Qualitative analysis of some systems of integral equations*, Calc. Var. Partial Differential Equations, **26** (2006), 447–457.

[21] T. Kilpelaiinen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137–161.

[22] N. Kawano, E. Yanagida, S. Yotsutani, *Structure theorems for positive radial solutions to* $div(|Du|^{m-2}Du)+K(|x|)u^q=0$ *in* R^n, J. Math. Soc. Japan, **45** (1993), 719–742.

[23] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111** (2002), 1–49.

[24] Y. Lei, *Decay rates for solutions of an integral system of Wolff type*, Potential Anal., **35** (2011), 387–402.

[25] Y. Lei, C. Li, *Integrability and asymptotics of positive solutions of a* γ*-Laplace system*, J. Differential Equations, **252** (2012), 2739–2758.

[26] Y. Lei, C. Li, C. Ma, *Decay estimation for positive solutions of a* γ*-Laplace equation*, Discrete Contin. Dyn. Syst., **30** (2011), 547–558.

[27] Y. Lei, C. Li, C. Ma, *Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system*, Calc. Var. Partial Differential Equations, **45** (2012), 43–61.

[28] C. Li, *Local asymptotic symmetry of singular solutions to nonlinear elliptic equations*, Invent. Math., **123** (1996), 221–231.

[29] C. Li, *A degree theory approach for the shooting method*, arXiv:1301.6232v1, 2013.

[30] C. Li, L. Ma, *Uniqueness of positive bound states to Schrödinger systems with critical exponents*, SIAM J. Math. Anal., **40**, (2008), 1049–1057.

[31] Y.-Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc., **6** (2004), 153–180.

[32] Y. Li, *Asymptotic behavior of positive solutions of equation* $\Delta u + K(x)u^p = 0$ *in* R^n, J. Differential equations, **95** (1992), 304–330.

[33] Y. Li, W.-M. Ni, *On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in* R^n *I. Asymptotic behavior*, Arch. Rational Mech. Anal., **118** (1992), 195–222.

[34] Y. Li, W.-M. Ni, *On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in* R^n *I. Radial symmetry*, Arch. Rational Mech. Anal., **118** (1992), 223–243.

[35] E. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math., **118** (1983), 349–374.

[36] C. Lin, *A classification of solutions of a conformally invariant fourth order equation in* R^n, Comm. Math. Helv., **73** (1998), 206–231.

[37] J. Liu, Y. Guo, Y. Zhang, *Liouville-type theorems for polyharmonic systems in* R^n, J. Differential equations, **225** (2006), 685–709.

[38] C. Ma, W. Chen, C. Li, *Regularity of solutions for an integral system of Wolff type*, Adv. Math., **226** (2011), 2676–2699.

[39] E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic systems in* R^n, Differential Integral equations, **9** (1996), 465–479.

[40] W.-M. Ni, J. Serrin, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case*, Accad. Naz. Lincei., **77** (1986), 231–257.

[41] M. Otani, *Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations*, J. Funct. Anal., **76** (1988), 140–159.

[42] L. A. Peletier, J. Serrin, *Ground states for the prescribed mean curvature equation*, Proc. Amer. Math. Soc., **100** (1987), 694–700.

[43] P. Pucci, J. Serrin, *A general variational identity*, Indiana Univ. J. Math., **35** (1986), 681–703.

[44] N. Phuc, I. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type*, Ann. of Math., **168** (2008), 859–914.

[45] J. Serrin, H. Zou, *Non-existence of positive solution of Lane-Emden systems*, Differential Integral equations, **9** (1996), 635–653.

[46] J. Serrin, H. Zou, *Existence of positive solutions of the Lane-Emden system*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), suppl., 369–380.

[47] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math.,**189** (2002), 79–142.

[48] P. Souplet, *The proof of the Lane-Emden conjecture in 4 space dimensions*, Adv. Math., **221** (2009), 1409–1427.

[49] S. Sun, Y. Lei, *Fast decay estimates for integrable solutions of the Lane-Emden type integral systems involving the Wolff potentials*, J. Funct. Anal., **263** (2012), 3857–3882.

[50] J. Wei, X. Xu, *Classification of solutions of higher order conformally invariant equations*, Math. Ann., **313** (1999), 207–228.

[51] X. Yu, *Liouville type theorems for integral equations and integral systems*, Calc. Var. Partial Differential Equations, **46** (2013), 75–95.

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